Runtime Analysis of Quantum Programs

A Formal Approach

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The role of quantum program resource analysis

Resource analysis is a particularly (more) relevant problem for quantum programs as it allows:

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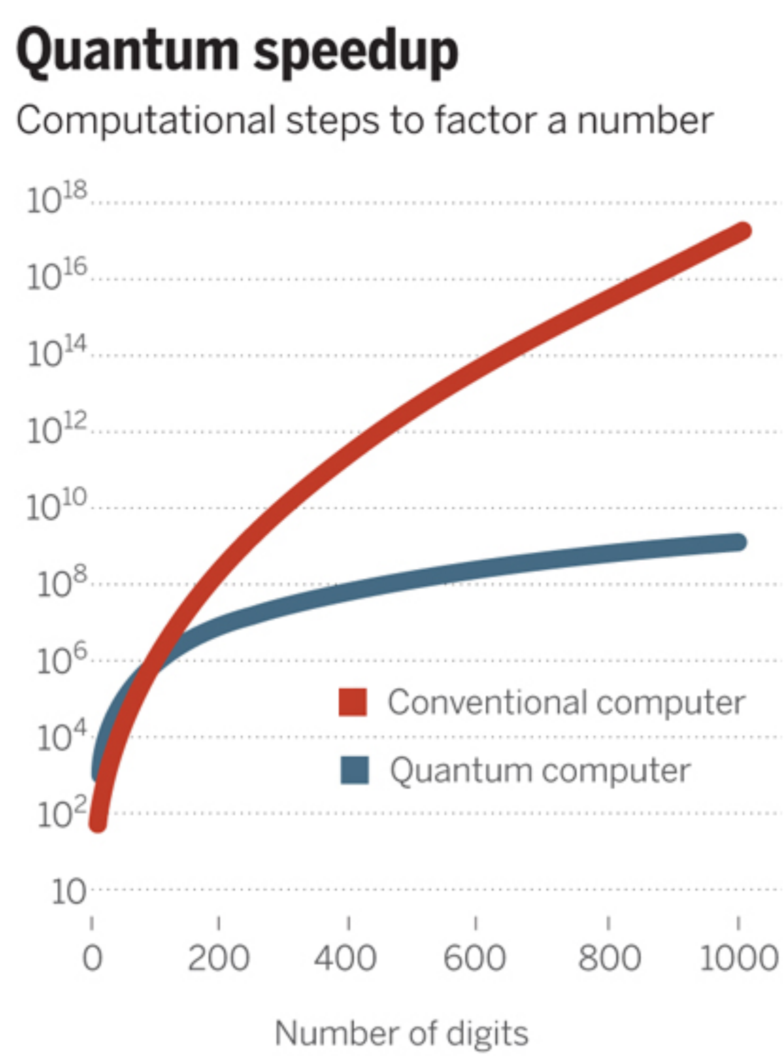
Resource analysis is a particularly (more) relevant problem for quantum programs as it allows:

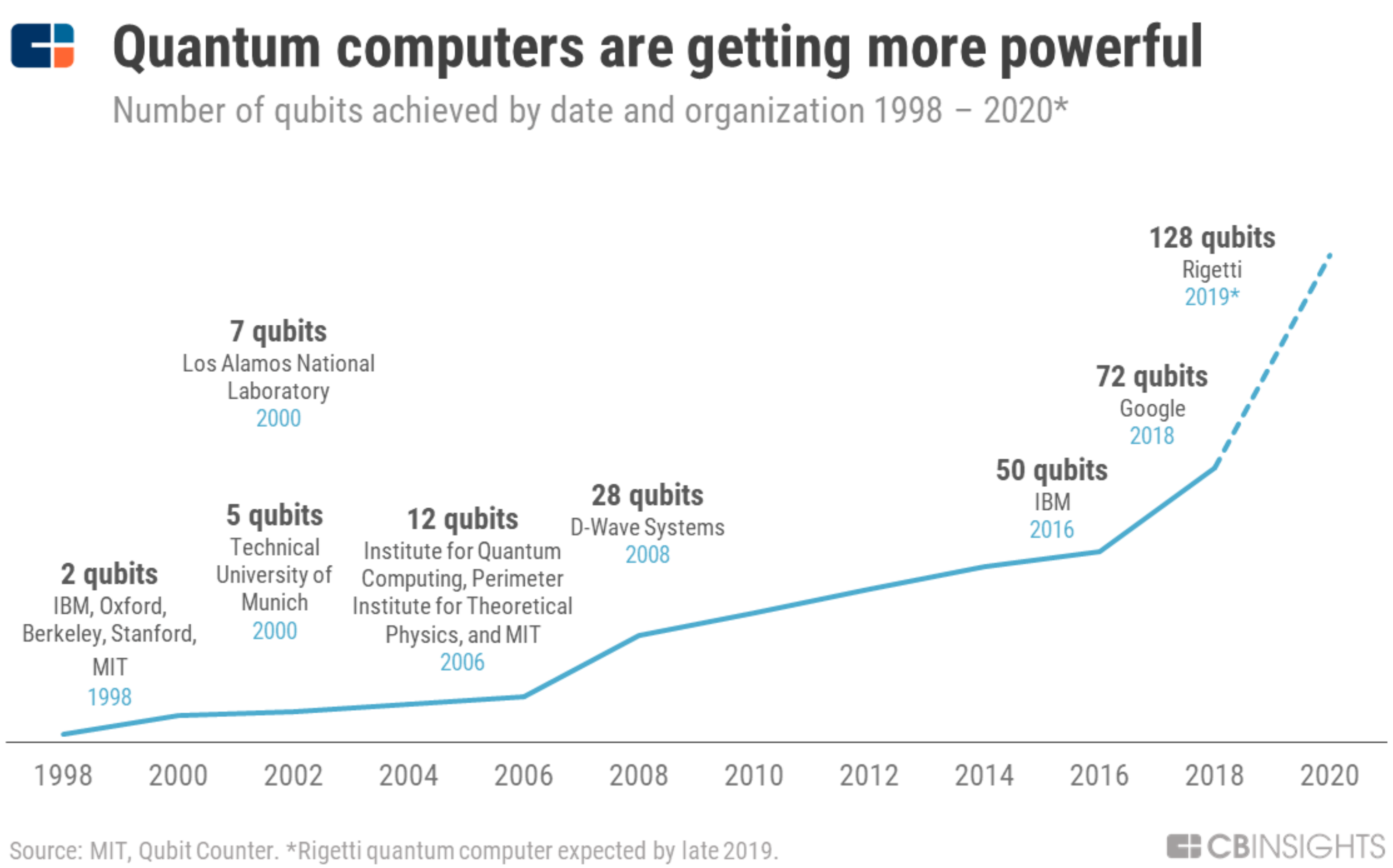
1. Validating the “effectivity” of the quantum computing model



The role of quantum program resource analysis

Resource analysis is a particularly (more) relevant problem for quantum programs as it allows:

2. Determining which quantum algorithms will be (shortly?) implementable in real quantum hardware



Current approach to quantum program resource analysis

[Quipper, Scaffold, LIQ*U*i|>]

1. Translate the (high-level) program into a (low-level) quantum circuit

```
#define n 1000
module foo(qbit q[n])
{
  for(int i=0;i<n;i++)
    H(q[i]);
  CNOT(q[n-1],q[0]);
}
module main()
{
  qbit b[n];
  foo(b);
}
```

```
qbit b[1000];
H ( b[0] );
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.
.
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2. Read off the **number of qubits and gates** in the circuit

Resources				
Qubit	X	Z	H	T
2	400	27800	54300	55100

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Severe limitations

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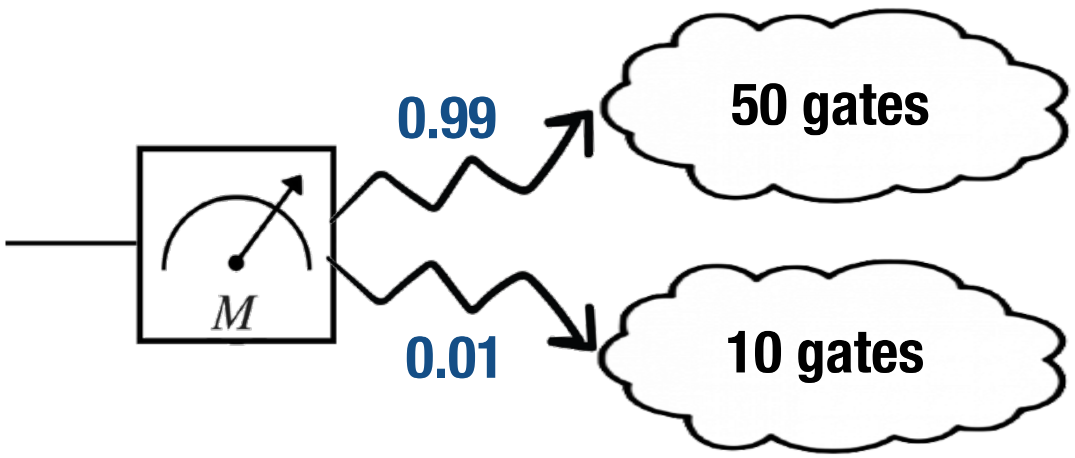

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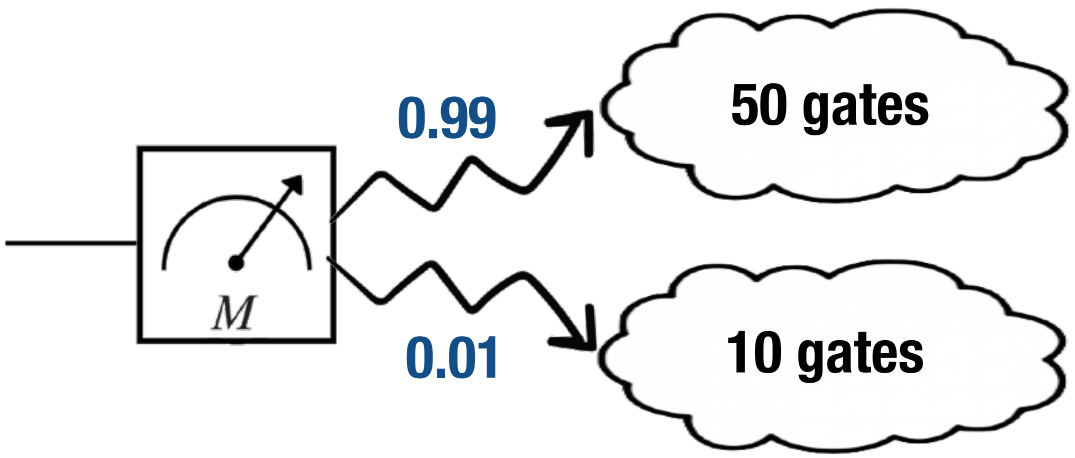

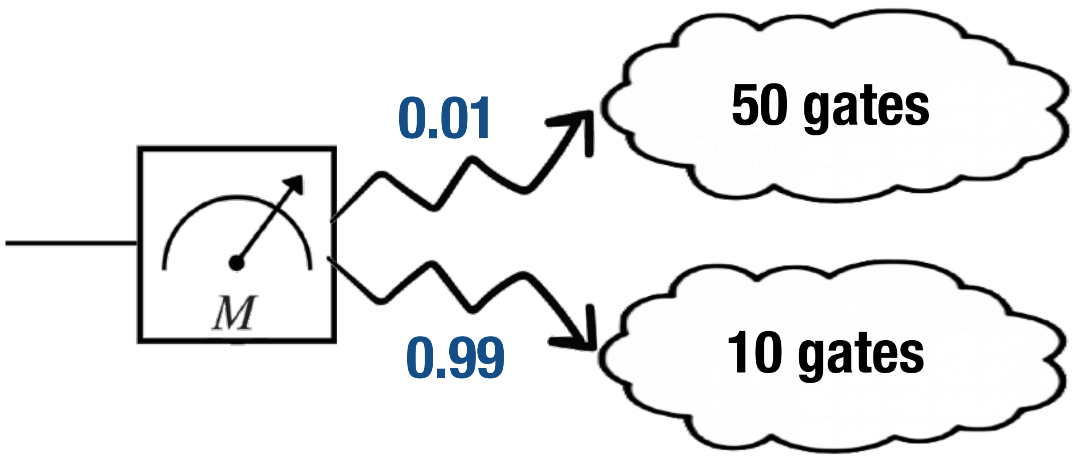

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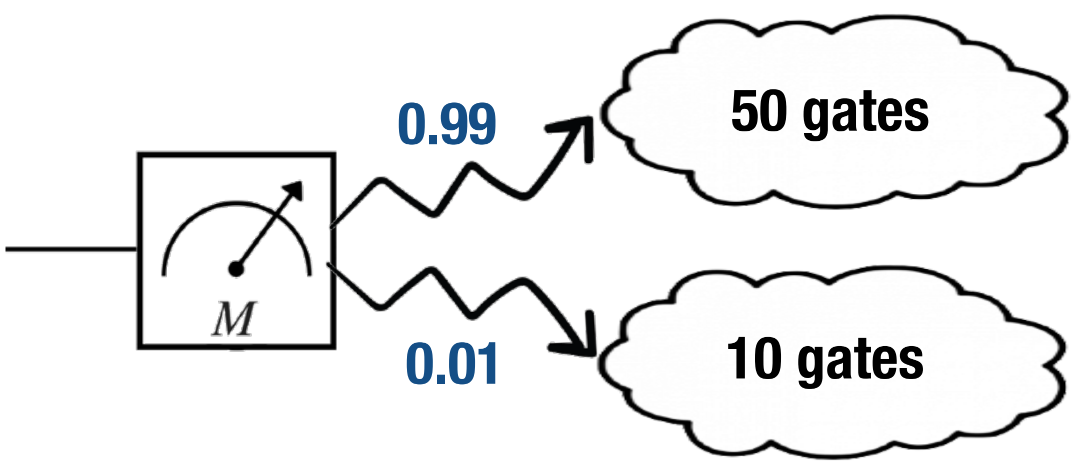

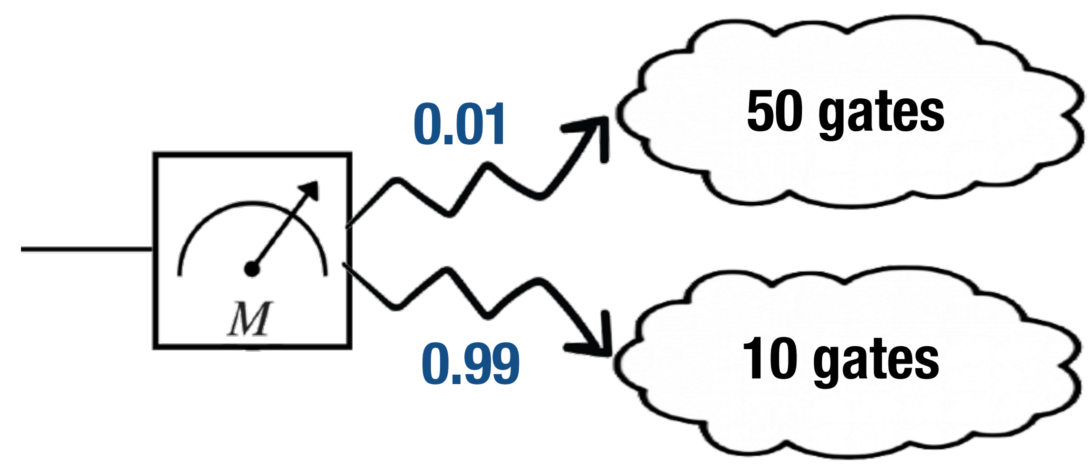

Same number of gates but (very!) different resource profiles

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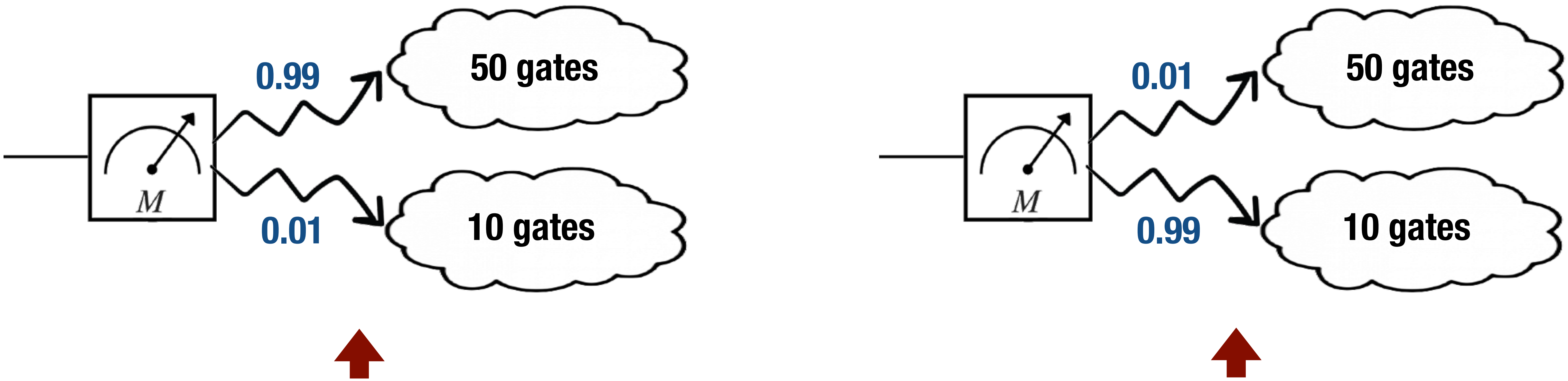
👎 Number of gates is a very poor resource measurement
(likelihood of execution paths is disregarded)



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```
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  for(int i=0;i<n;i++)
                        prob. < 1
   ...
}
```

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Sample algorithms out of scope

- BB84 quantum key distribution algorithm
- Simon's algorithm

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- 👎 Rigid and low-level cost model

Our contribution

Calculus à la weakest precondition for reasoning about the runtime of quantum programs

- **Flexible:** accommodates multiple runtime models
- **Sensible:** accounts for execution probabilities
- **Expressive:** applies to programs with unbounded loops

Based on existing techniques for probabilistic programs
[Kaminski, Katoen, Matheja & Olmedo - ESOP'16, LICS'16, JACM 65:5]

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Core imperative language over quantum variables with classical control flow

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\begin{array}{lll}
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		while $(\mathcal{M}[\overline{q}] = 1)$ do c	quantum loop
		skip	no-op
		$c_1 ; c_2$	sequential composition

Our approach to the runtime of quantum programs

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set of program states

GOAL:

$$[\![c]\!]^{⧗} : \mathcal{S} \to \mathbb{R}_{\geq 0}^{\infty}$$

$$[\![c]\!]^{⧗}\, \rho \quad = \quad \text{runtime of } c \text{ from initial state } \rho$$

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APPROACH: Continuation passing style through **runtime transformer**

$$\mathsf{ert}\,[c] : \ (\mathcal{S} \to \mathbb{R}_{\geq 0}^{\infty}) \to (\mathcal{S} \to \mathbb{R}_{\geq 0}^{\infty})$$

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runtime of the program following c

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In particular,

$$\mathrm{ert}\,[c]\,(\lambda \rho'.\, 0) \quad = \quad [\![c]\!]^{\text{⧗}}$$

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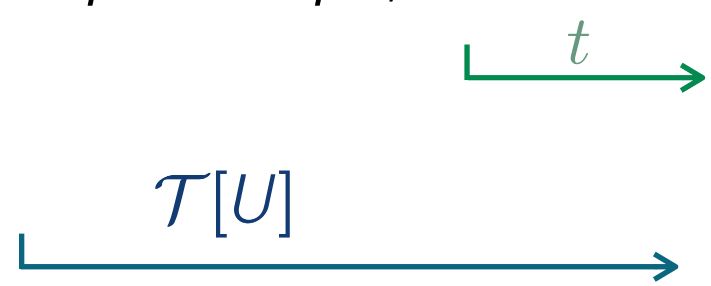

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$$\lambda\rho.\; U\rho U^{\dagger}$$

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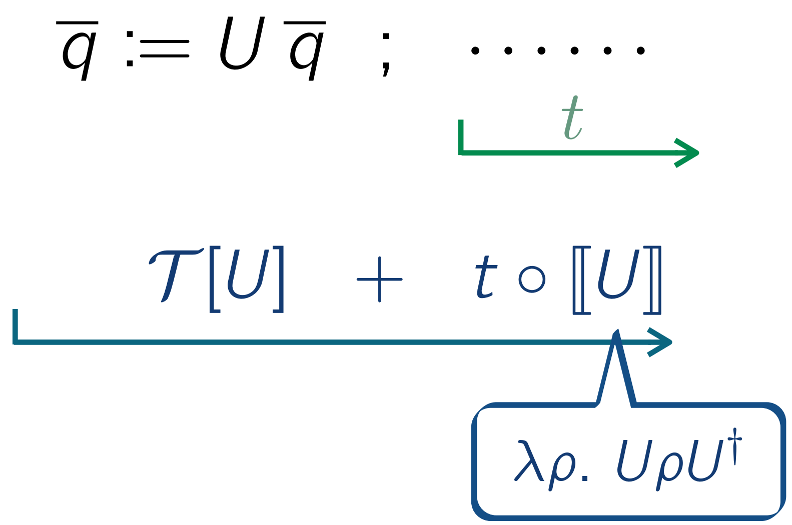

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We can establish upper bounds for the runtime of loops using a notion of **loop invariant**:

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$$\frac{F_t^{\langle \mathcal{M}, c\rangle}(I) \;\preceq\; I}{\text{ert}[\text{while } (\mathcal{M} = 1) \text{ do } c](t) \;\preceq\; I}$$

← *I* is a loop invariant

← *I* is an upper bound of the loop runtime

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GOAL: securely create and distribute a shared (symmetric) key between two parties.

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\\ initialize counter
k := |0⟩;
\\ while not reached m bits
while (M[k] = 1) do
   \\ flip Alice's and Bob's coins
   A := |++⟩; B := |+⟩;
   \\ measure Alice's coins
   □ · M_A[A] =
               \\ measure Bob's coin
     |eb⟩ → □ · M_B[B]
               \\ if Alice's and Bob's basis agree
               \\ store bit b and increment counter
               |e⟩ → k, Q := U_{P_b}[k, Q];
                          k := U_>[k];
               \\ if Alice's and Bob's basis disagree
               \\ discard bit b
               |¬e⟩ → skip
```

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   □ · M_A[A] =
            \\ measure Bob's coin
     |eb⟩ → □ · M_B[B]
              \\ if Alice's and Bob's basis agree
              \\ store bit b and increment counter
              |e⟩ → k, Q := U_{P_b}[k, Q];
                    k := U_>[k];
              \\ if Alice's and Bob's basis disagree
              \\ discard bit b
              |¬e⟩ → skip
```

Average time required to generate a key of *m* bits:

$$\mathcal{T}[|0\rangle] + 2\mathcal{T}m + \mathcal{T}[\mathcal{M}] \quad \in \mathcal{O}(m)$$

Case study: BB84 quantum key distribution algorithm

GOAL: securely create and distribute a shared (symmetric) key between two parties.

\\\\ *initialize counter*
$k := |0\rangle;$
\\\\ *while not reached m bits*
while $(\mathcal{M}[k] = 1)$ do
\\\\ *flip Alice's and Bob's coins*
$A := |{+}{+}\rangle;\ B := |{+}\rangle;$
\\\\ *measure Alice's coins*
$\square \cdot \mathcal{M}_A[A]\ =$
\\\\ *measure Bob's coin*
$|eb\rangle \rightarrow \square \cdot \mathcal{M}_B[B]$
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\\\\ *store bit b and increment counter*
$|e\rangle \rightarrow k, Q := U_{P_b}[k, Q];$
$k := U_{>}[k];$
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$|\neg e\rangle \rightarrow$ skip

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counter *k* initialization

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GOAL: securely create and distribute a shared (symmetric) key between two parties.

```
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k := |0⟩;
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   \\ measure Alice's coins
   □ · M_A[A] =
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loop body

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Final measurement upon loop exit

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      \\ if Alice's and Bob's basis agree
      \\ store bit b and increment counter
      $|e\rangle \rightarrow k, Q := U_{P_b}[k, Q];$
        $k := U_{\succ}[k];$
      \\ if Alice's and Bob's basis disagree
      \\ discard bit b
      $|\neg e\rangle \rightarrow$ skip

Average time required to generate a key of *m* bits:

$$\mathcal{T}[|0\rangle] + 2\mathcal{T}m + \mathcal{T}[\mathcal{M}] \quad \in \mathcal{O}(m)$$

counter *k* initialization

loop body

Final measurement upon loop exit

$$\mathcal{T} \ = \ \mathcal{T}[\mathcal{M}] + \mathcal{T}[|{+}{+}\rangle] + \mathcal{T}[|{+}\rangle] + \mathcal{T}[\mathcal{M}_A] + \mathcal{T}[\mathcal{M}_B] + \tfrac{1}{2}\left(\tfrac{1}{2}\mathcal{T}[U_{P_0}] + \tfrac{1}{2}\mathcal{T}[U_{P_1}] + \mathcal{T}[U_{\succ}]\right) + \tfrac{1}{2}$$

Conclusion

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First step to a *formal and compelling* resource analysis of quantum programs

- Existing techniques for probabilistic programs extend smoothly to quantum programs

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Future work

- Connection to an operational model
- Language extensions
- Automation

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Future work

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Thanks!

BACKUP SLIDES

Language semantics

$$
\begin{aligned}
[\![\mathrm{skip}]\!](\rho) &= \rho \\
[\![q := |b\rangle]\!](\rho) &= \rho[q \mapsto |b\rangle] \\
[\![\overline{q} := U\ \overline{q}]\!](\rho) &= U^{\uparrow q} \rho {U^{\uparrow q}}^{\dagger} \\
[\![c_1 ; c_2]\!](\rho) &= [\![c_1]\!]([\![c_2]\!](\rho)) \\
[\![\square\ \mathcal{M}[\overline{q}]\ =\ \overline{m \to c_m}]\!](\rho) &= \textstyle\sum_m \mathrm{Pr}_\rho[\mathcal{M}{=}m] \cdot [\![c_m]\!](\rho|_{\mathcal{M}=m}) \\
[\![\mathrm{while}\ (\mathcal{M}[\overline{q}] = 1)\ \mathrm{do}\ c]\!](\rho) &= \mathit{lfp}\big(\Phi^{\langle \mathcal{M}, c\rangle}\big)
\end{aligned}
$$

$\mathrm{tr}({M_m}^{\dagger} M_m \rho)$

$\dfrac{M_m \rho {M_m}^{\dagger}}{\mathrm{tr}({M_m}^{\dagger} M_m \rho)}$

$\lambda X.\ \lambda \rho'.\ \mathrm{Pr}_{\rho'}[\mathcal{M}{=}0] \cdot \rho'|_{\mathcal{M}=0} + X\big(\mathrm{Pr}_{\rho'}[\mathcal{M}{=}1] \cdot [\![c]\!](\rho'|_{\mathcal{M}=1})\big)$

Full definition of ert transformer

$$
\begin{aligned}
\text{ert}[\text{skip}](t) &= \lambda\rho.\ 1 + t(\rho) \\
\text{ert}[q := |b\rangle](t) &= \lambda\rho.\ \mathcal{T}[|b\rangle] + t\big(\rho[q \mapsto |b\rangle]\big) \\
\text{ert}[\overline{q} := U\ \overline{q}](t) &= \lambda\rho.\ \mathcal{T}[U] + t\Big(U^{\uparrow q}\rho {U^{\uparrow q}}^{\dagger}\Big) \\
\text{ert}[c_1; c_2](t) &= \text{ert}[c_1](\text{ert}[c_2](t)) \\
\text{ert}[\square\ \mathcal{M}[\overline{q}]\ =\ \overline{m \to c_m}](t) &= \lambda\rho.\ \mathcal{T}[\mathcal{M}] + \textstyle\sum_m \text{Pr}_\rho[\mathcal{M}{=}m] \cdot \text{ert}[c_m](t)\big(\rho|_{\mathcal{M}=m}\big) \\
\text{ert}[\text{while } (\mathcal{M}[\overline{q}] = 1) \text{ do } c](t) &= \mathit{lfp}\big(F_t^{\langle \mathcal{M}[\overline{q}], c\rangle}\big)
\end{aligned}
$$

$$\lambda t'.\ \lambda\rho.\ \mathcal{T}[\mathcal{M}] + \text{Pr}_\rho[\mathcal{M}{=}1] \cdot \text{ert}[c](t')(\rho|_{\mathcal{M}=1}) + \text{Pr}_\rho[\mathcal{M}{=}0] \cdot t(\rho|_{\mathcal{M}=0})$$