Weakest Precondition Reasoning for Expected Run–Times of Probabilistic Programs

Benjamin Kaminski Joost-Pieter Katoen

Christoph Matheja Federico Olmedo

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

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- Model probability distributions in machine learning

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Better Question:

What is the expected run–time (ERT) of C on input σ?

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- ERT of C can be infinite, even if C terminates almost–surely[1]

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[1] i.e. with probability 1

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- Complete partial order on $\mathbb{T}$:

$$t_1 \preceq t_2 \quad \text{iff} \quad \forall\, \sigma \in \Sigma\colon t_1(\sigma) \leq t_2(\sigma)$$

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ERT in Terms of ert

$$\mathsf{ert}\,[C]\,(\mathbf{0})\,(\sigma) \;=\; \text{“ERT of } C \text{ on input } \sigma\text{”}$$

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  - ert is sound and complete with respect to Nielson's logic

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- Nielson's Hoare–style logic for reasoning about run–time orders of magnitude of *deterministic programs*:
  - Nielson's logic relies on introducing additional *logical* variables
  - ert is sound and complete with respect to Nielson's logic
  - ert calculus is arguably easier to apply — no additional variables!

Case Study: The Coupon Collector's Problem

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- The coupon collector is a well–known problem

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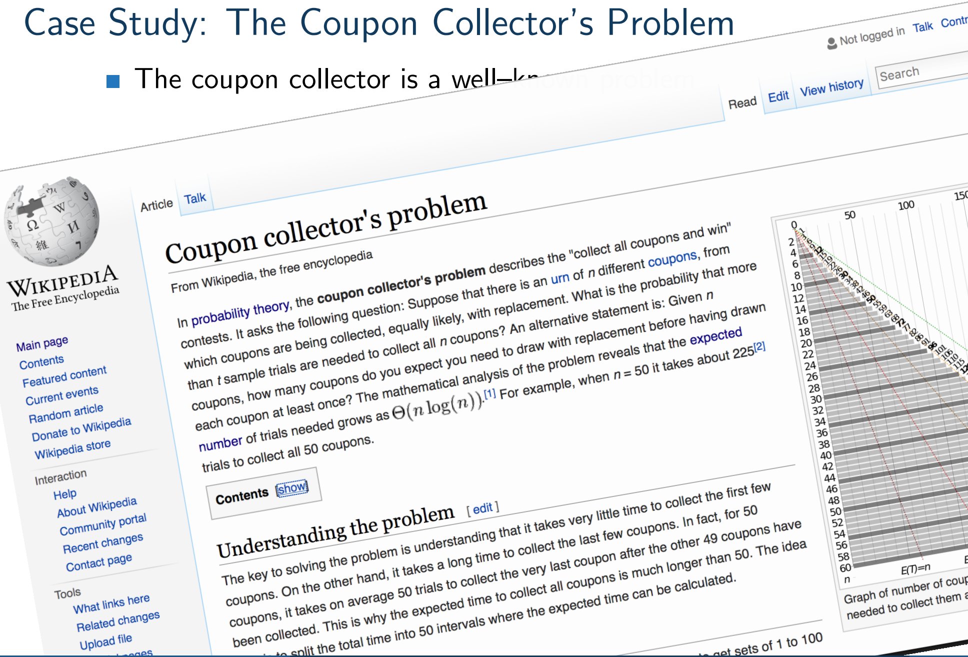

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Coupon collector's problem

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In probability theory, the **coupon collector's problem** describes the "collect all coupons and win" contests. It asks the following question: Suppose that there is an urn of *n* different coupons, from which coupons are being collected, equally likely, with replacement. What is the probability that more than *t* sample trials are needed to collect all *n* coupons? An alternative statement is: Given *n* coupons, how many coupons do you expect you need to draw with replacement before having drawn each coupon at least once? The mathematical analysis of the problem reveals that the expected number of trials needed grows as $\Theta(n \log(n))$.[1] For example, when $n = 50$ it takes about 225[2] trials to collect all 50 coupons.

Contents [show]

Understanding the problem [edit]

The key to solving the problem is understanding that it takes very little time to collect the first few coupons. On the other hand, it takes a long time to collect the last few coupons. In fact, for 50 coupons, it takes on average 50 trials to collect the very last coupon after the other 49 coupons have been collected. This is why the expected time to collect all coupons is much longer than 50. The idea ... to split the total time into 50 intervals where the expected time can be calculated.

... get sets of 1 to 100

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Main page
Contents
Featured content
Current events
Random article
Donate to Wikipedia
Wikipedia store

Interaction
Help
About Wikipedia
Community portal
Recent changes
Contact page

Tools
What links here
Related changes
Upload file

Graph of number of coup... needed to collect them a...

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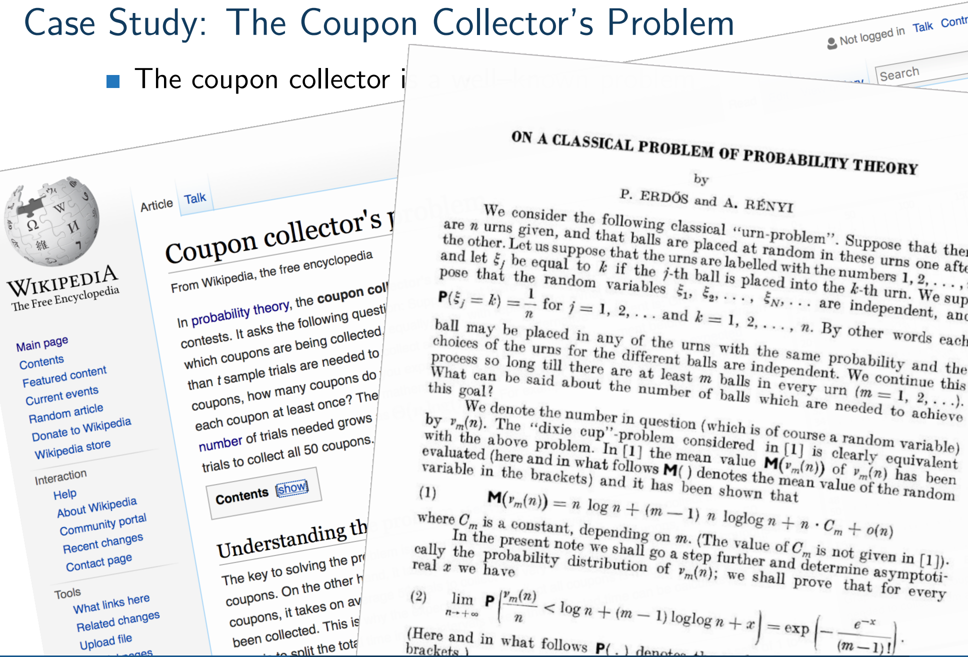

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Search

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The Free Encyclopedia

Main page
Contents
Featured content
Current events
Random article
Donate to Wikipedia
Wikipedia store

Interaction
Help
About Wikipedia
Community portal
Recent changes
Contact page

Tools
What links here
Related changes
Upload file

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Contents [show]

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ON A CLASSICAL PROBLEM OF PROBABILITY THEORY

by

P. ERDŐS and A. RÉNYI

We consider the following classical "urn-problem". Suppose that ther are n urns given, and that balls are placed at random in these urns one afte the other. Let us suppose that the urns are labelled with the numbers $1, 2, \ldots,$ and let ξ_j be equal to k if the j-th ball is placed into the k-th urn. We sup pose that the random variables $\xi_1, \xi_2, \ldots, \xi_N, \ldots$ are independent, and $\mathbf{P}(\xi_j = k) = \frac{1}{n}$ for $j = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$. By other words each ball may be placed in any of the urns with the same probability and the choices of the urns for the different balls are independent. We continue this process so long till there are at least m balls in every urn ($m = 1, 2, \ldots$). What can be said about the number of balls which are needed to achieve this goal?

We denote the number in question (which is of course a random variable) by $\nu_m(n)$. The "dixie cup"-problem considered in [1] is clearly equivalent with the above problem. In [1] the mean value $\mathbf{M}(\nu_m(n))$ of $\nu_m(n)$ has been evaluated (here and in what follows $\mathbf{M}(\ )$ denotes the mean value of the random variable in the brackets) and it has been shown that

$$\mathbf{M}(\nu_m(n)) = n \log n + (m-1)\, n \log\log n + n \cdot C_m + o(n) \tag{1}$$

where C_m is a constant, depending on m. (The value of C_m is not given in [1]).

In the present note we shall go a step further and determine asymptotically the probability distribution of $\nu_m(n)$; we shall prove that for every real x we have

$$\lim_{n\to+\infty} \mathbf{P}\left(\frac{\nu_m(n)}{n} < \log n + (m-1)\log\log n + x\right) = \exp\left(-\frac{e^{-x}}{(m-1)!}\right). \tag{2}$$

(Here and in what follows $\mathbf{P}(\ldots)$ denotes the ... brackets.)

Case Study: The Coupon Collector's Problem

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\begin{array}{l}
cp \texttt{ := } [0, \ldots, 0];\ i \texttt{ := } 1;\ x \texttt{ := } N;\\
\texttt{while}\,(x > 0)\,\{\\
\qquad \texttt{while}\,(cp[i] \neq 0)\,\{\ i \texttt{ :}\!\approx \texttt{Unif}[1 \ldots N]\ \};\\
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- Harmonic number $\mathcal{H}_{N-1}$ is in $\Theta(\log N)$
- Coupon collector program runs in $\Theta(N \cdot \log N)$ for $N > 0$

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Thank you for your kind attention!

Backup Slides: The Actual Rule for Assignments

C	$\mathsf{ert}\,[C]\,(t)$
$x :\approx \mu$	$\mathbf{1} + \lambda\sigma_\bullet\ \mathsf{E}_{[\![\mu]\!](\sigma)}\,(\lambda v.\, t\,[x/v]\,(\sigma))$

Backup Slides: ert Calculations and Proof Rule Application

Example 4 (Geometric distribution). Consider loop

$$C_{\mathtt{geo}}\colon\quad \mathtt{while}\,(c=1)\,\{c :\approx {}^1\!/_2\cdot\langle 0\rangle + {}^1\!/_2\cdot\langle 1\rangle\}\ .$$

From the calculations below we conclude that $I = \mathbf{1} + [\![c=1]\!]\cdot\mathbf{4}$ is an upper invariant with respect to $\mathbf{0}$:

$$\begin{aligned}
&\mathbf{1} + [\![c\neq 1]\!]\cdot\mathbf{0} + [\![c=1]\!]\cdot \mathsf{ert}\,[c :\approx {}^1\!/_2\cdot\langle 0\rangle + {}^1\!/_2\cdot\langle 1\rangle]\,(I)\\
&\quad = \ \mathbf{1} + [\![c=1]\!]\cdot\big(\mathbf{1} + \tfrac{1}{2}\cdot I\,[c/0] + \tfrac{1}{2}\cdot I\,[c/1]\big)\\
&\quad = \ \mathbf{1} + [\![c=1]\!]\cdot\big(\mathbf{1} + \tfrac{1}{2}\cdot \underbrace{(\mathbf{1} + [\![0=1]\!]\cdot\mathbf{4})}_{=\mathbf{1}} + \tfrac{1}{2}\cdot \underbrace{(\mathbf{1} + [\![1=1]\!]\cdot\mathbf{4})}_{=\mathbf{5}}\big)\\
&\quad = \ \mathbf{1} + [\![c=1]\!]\cdot\mathbf{4} \ = \ I \ \preceq \ I
\end{aligned}$$

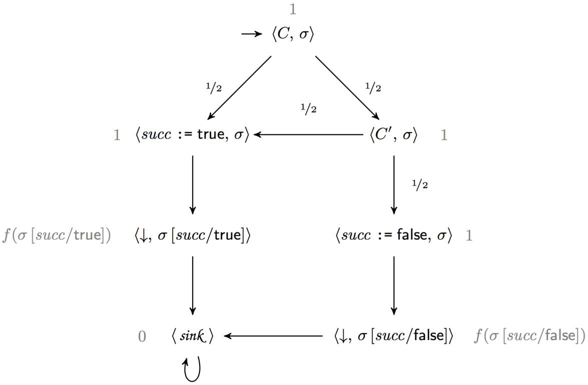
Then applying Theorem 3 we obtain

$$\mathsf{ert}\,[C_{\mathtt{geo}}]\,(\mathbf{0}) \ \preceq \ \mathbf{1} + [\![c=1]\!]\cdot\mathbf{4}\ .$$

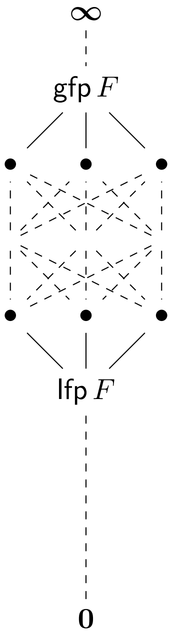
In words, the expected run–time of $C_{\mathtt{geo}}$ is at most 5 from any initial state where $c=1$ and at most 1 from the remaining states. △

Backup Slides: Operational RMDP

$$C_{trunc}\colon\quad \texttt{if}\ \left({}^1\!/_2\cdot\langle\mathsf{true}\rangle + {}^1\!/_2\cdot\langle\mathsf{false}\rangle\right)\ \{\mathit{succc} := \mathsf{true}\}\ \texttt{else}\ \{$$
$$\texttt{if}\ \left({}^1\!/_2\cdot\langle\mathsf{true}\rangle + {}^1\!/_2\cdot\langle\mathsf{false}\rangle\right)\ \{\mathit{succ} := \mathsf{true}\}$$
$$\texttt{else}\ \{\mathit{succ} := \mathsf{false}\}$$
$$\}$$



Backup Slides: Park's Lemma



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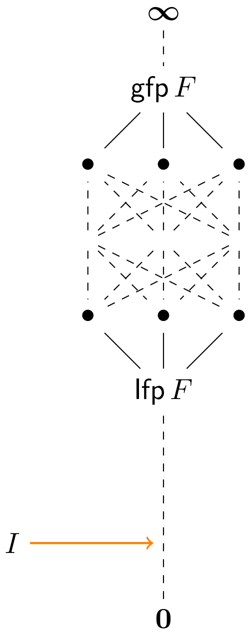

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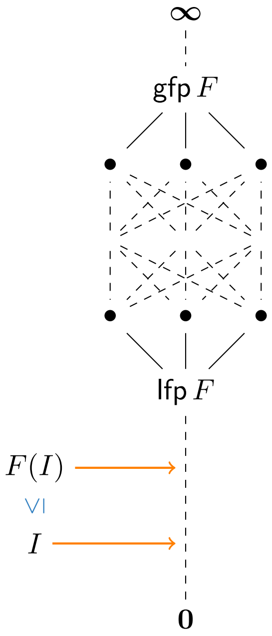

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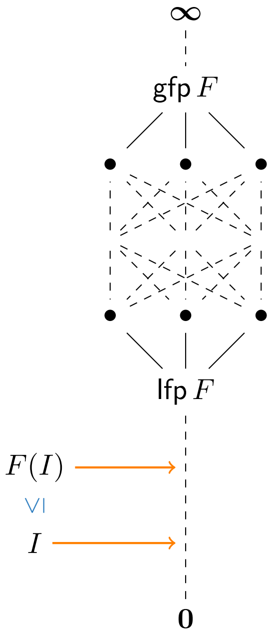
$F(I) \leq I$ implies $\mathsf{lfp}\, F \leq I$



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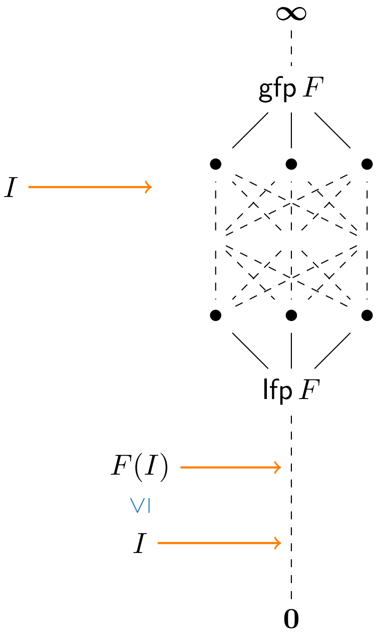

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