Probabilistic Relational Reasoning for Differential Privacy

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Federico Olmedo Santiago Zanella Béguelin

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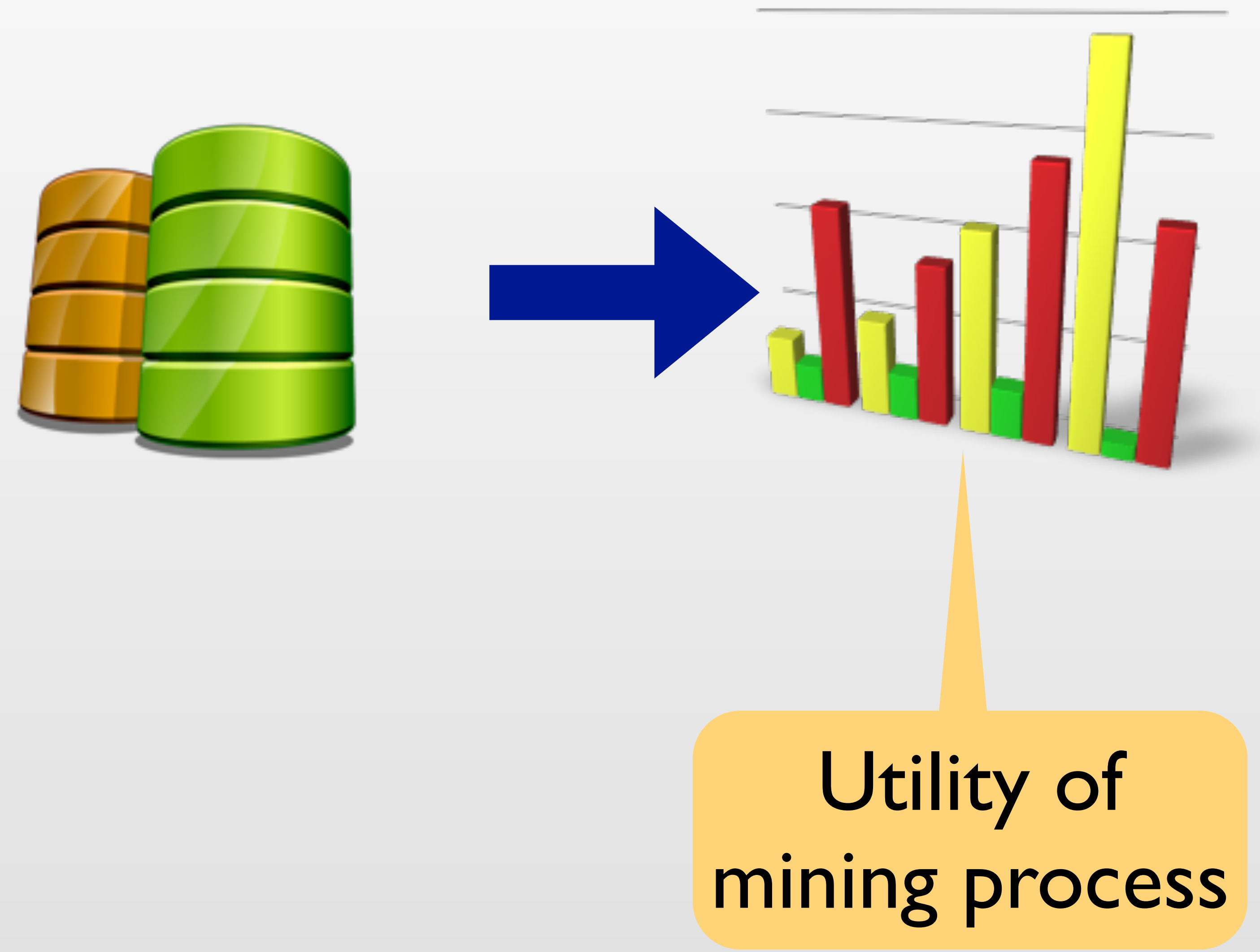
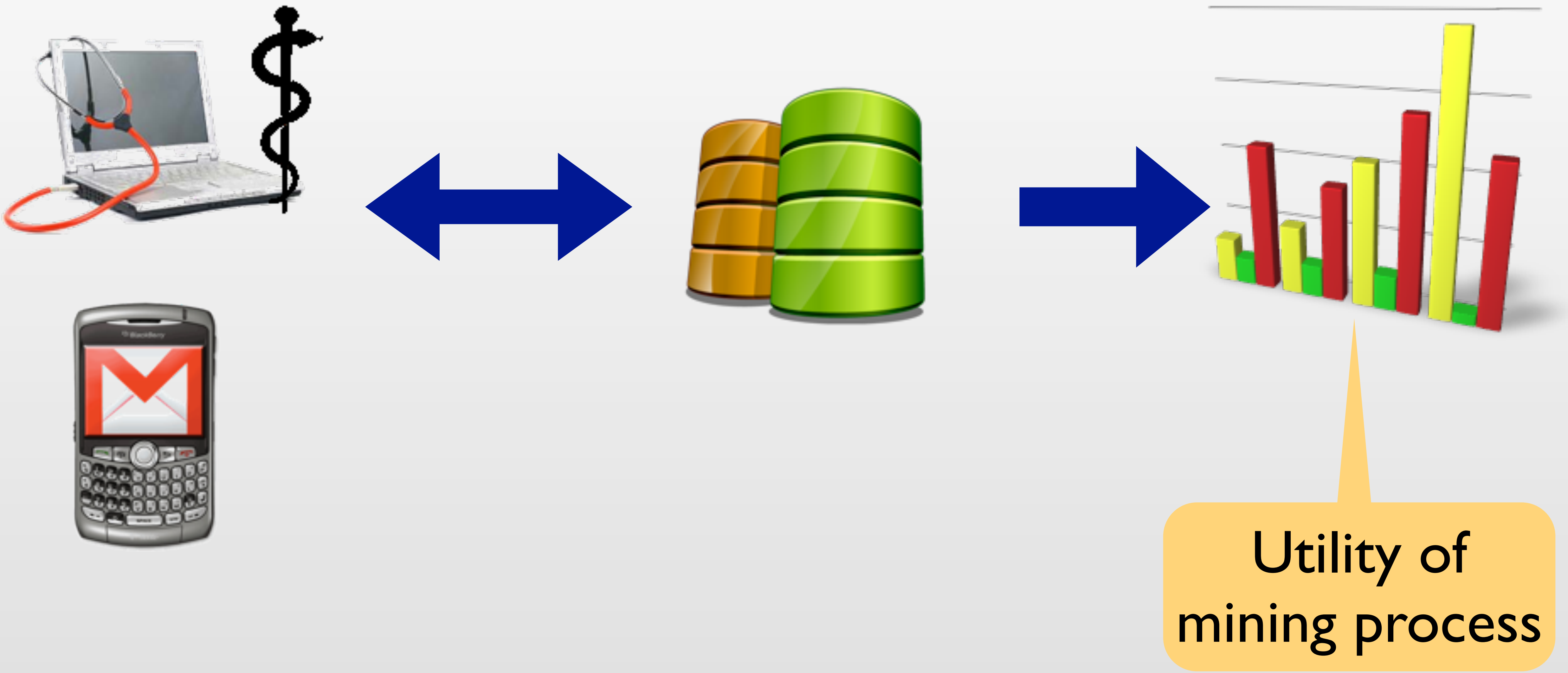
POPL 2012


Utility of
mining process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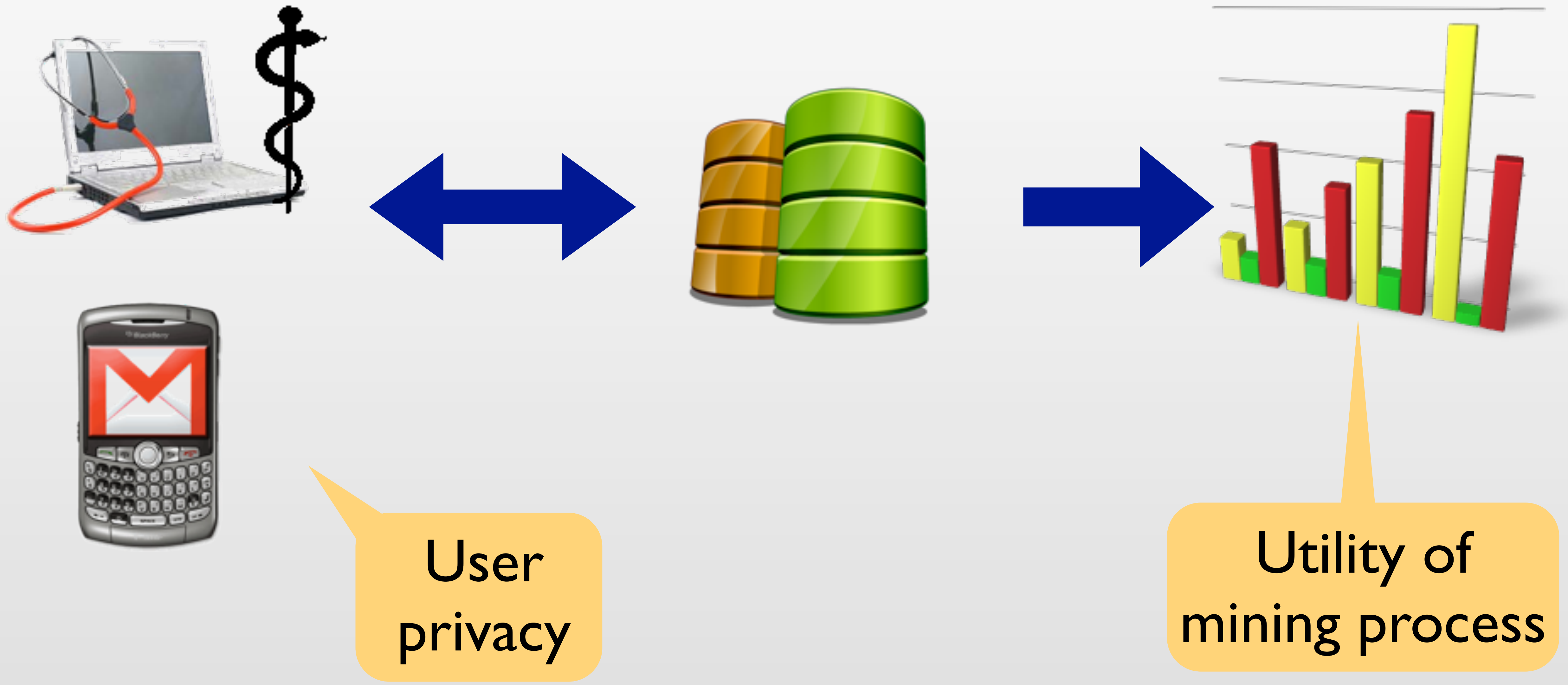
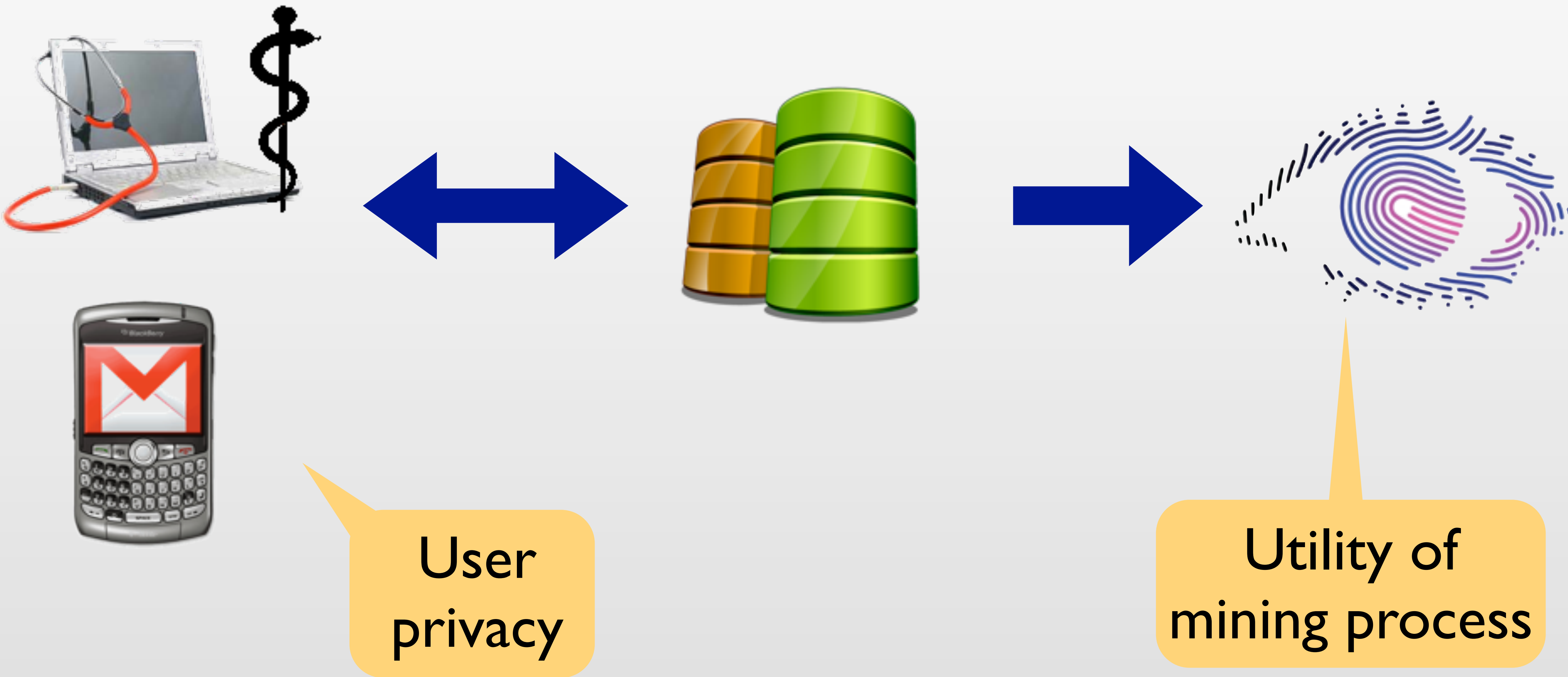

User privacy
Utility of mining process


User privacy
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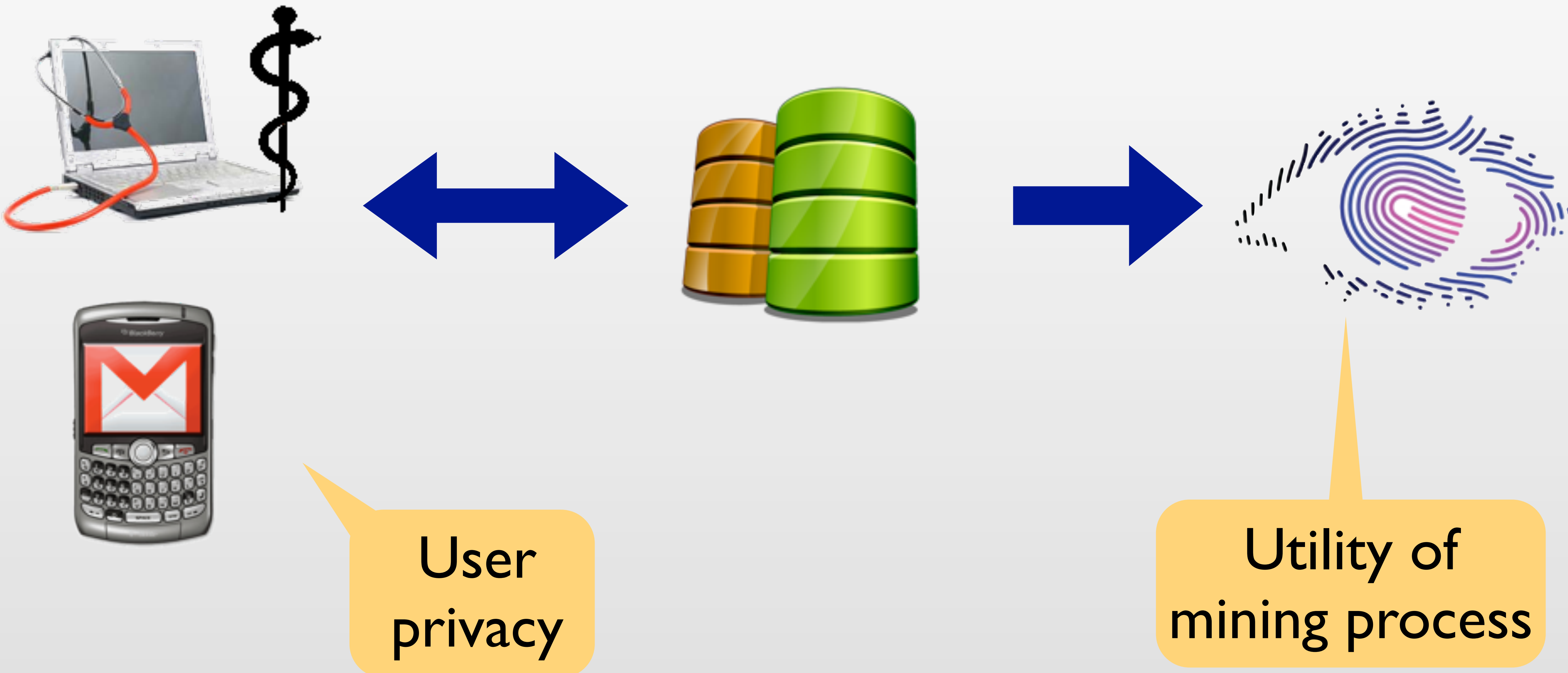
Conflicting requirements!



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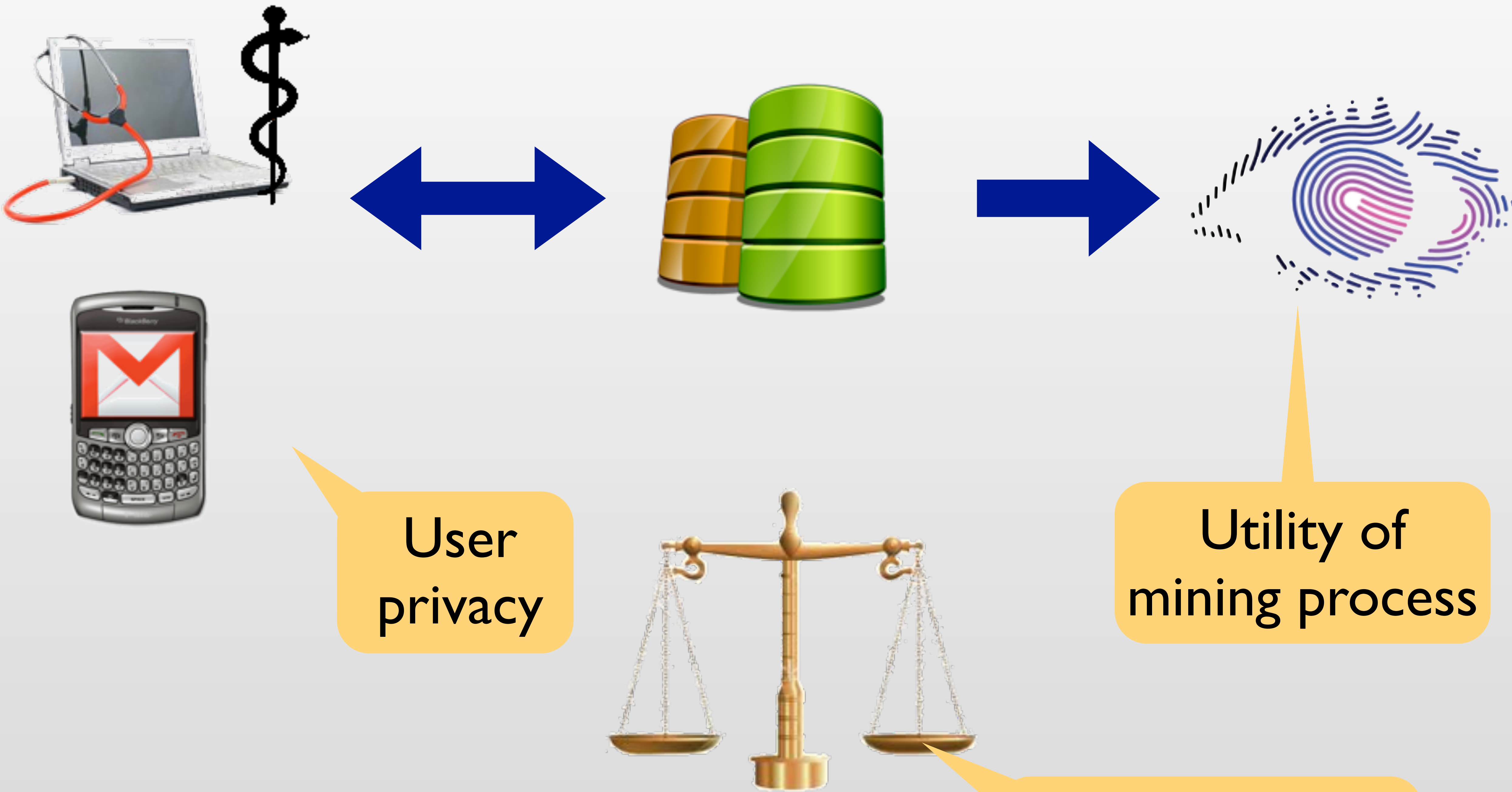


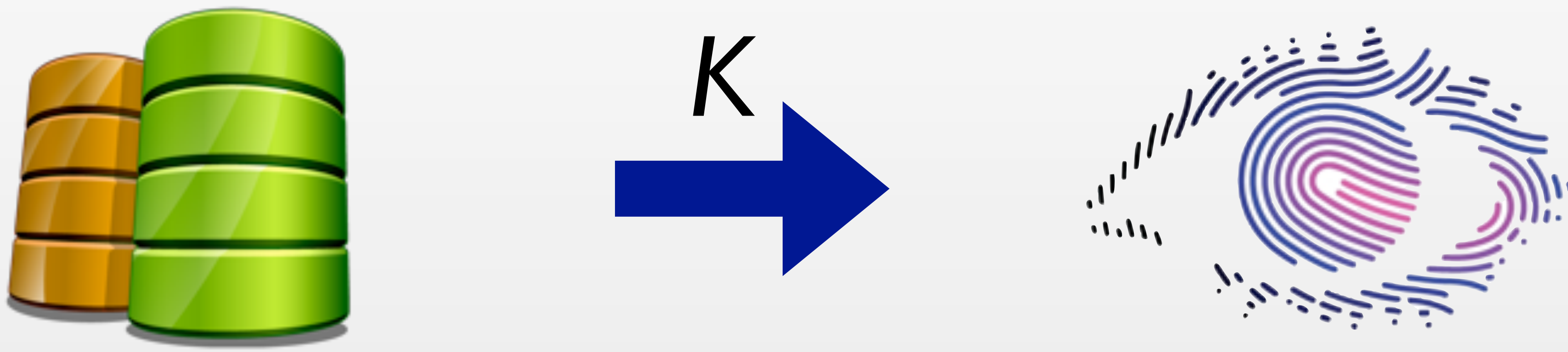
User privacy

Utility of mining process

Need to achieve flexible balance

Differential Privacy

Dwork [ICALP'06]



Differential Privacy

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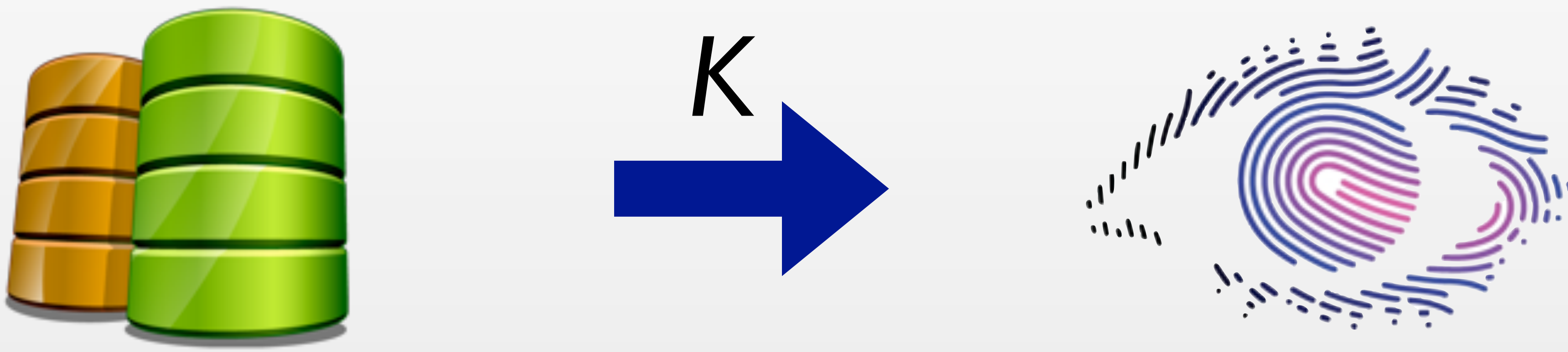

- Fix a (symmetric) adjacency relation Φ on databases

Differential Privacy

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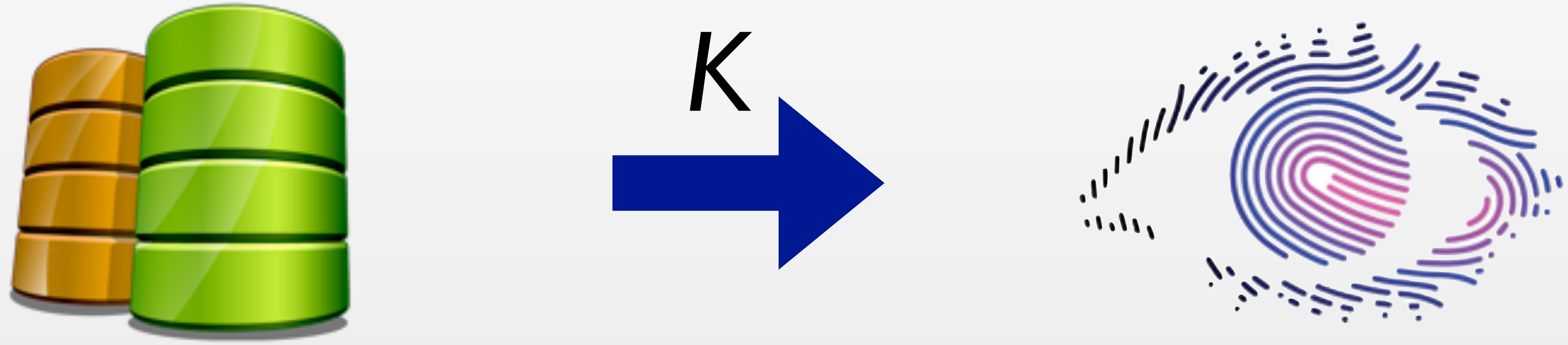

- Fix a (symmetric) adjacency relation Φ on databases
- Fix a privacy budget ε

Differential Privacy

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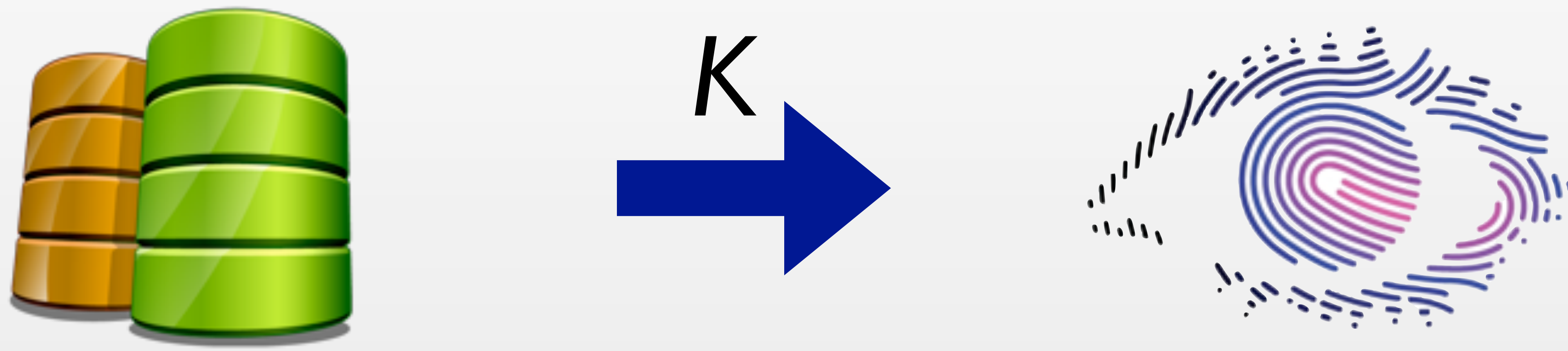


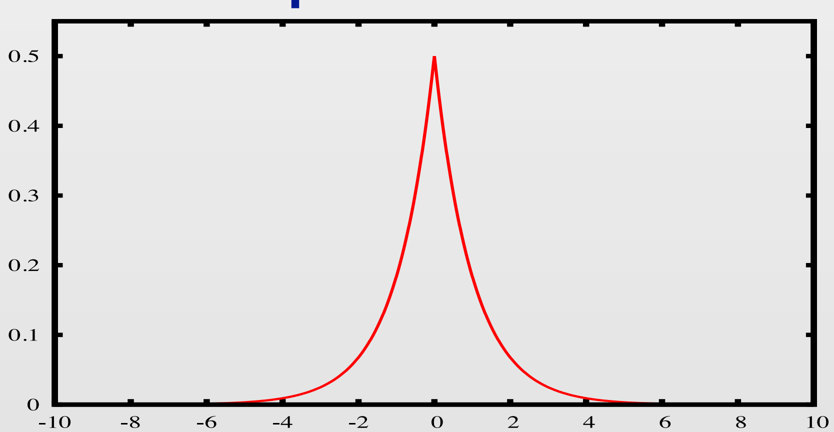
- Fix a (symmetric) adjacency relation Φ on databases
- Fix a privacy budget ε

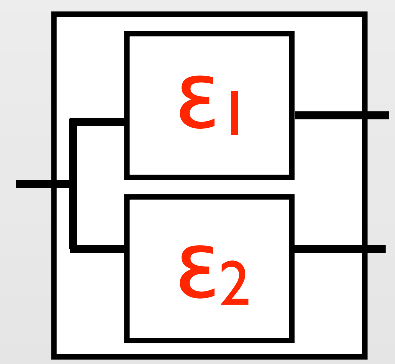
A randomized algorithm *K* is ε-differentially private w.r.t. Φ iff, for all databases D_1 and D_2, and events S

$$\Phi(D_1, D_2) \Longrightarrow \Pr[K(D_1) \in S] \leq \exp(\epsilon) \times \Pr[K(D_2) \in S]$$

Differential Privacy Primer

- Fundamentals
  - Laplacian mechanism
  - Composition theorems





$\varepsilon_1 + \varepsilon_2$

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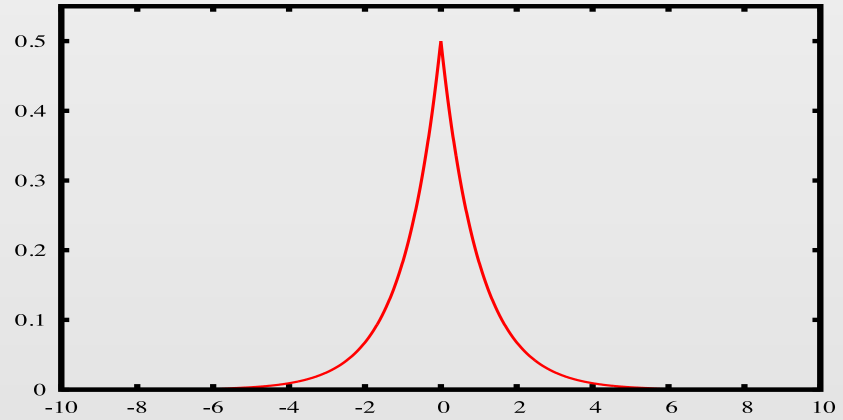

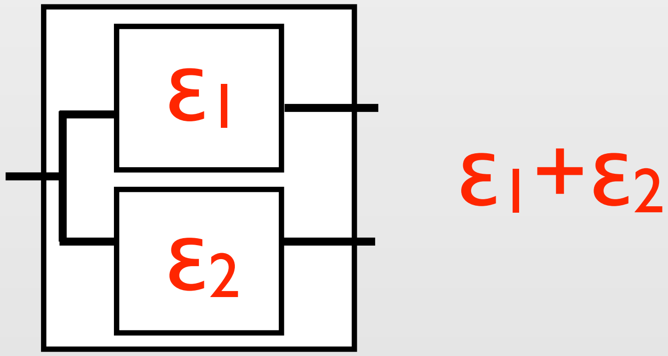

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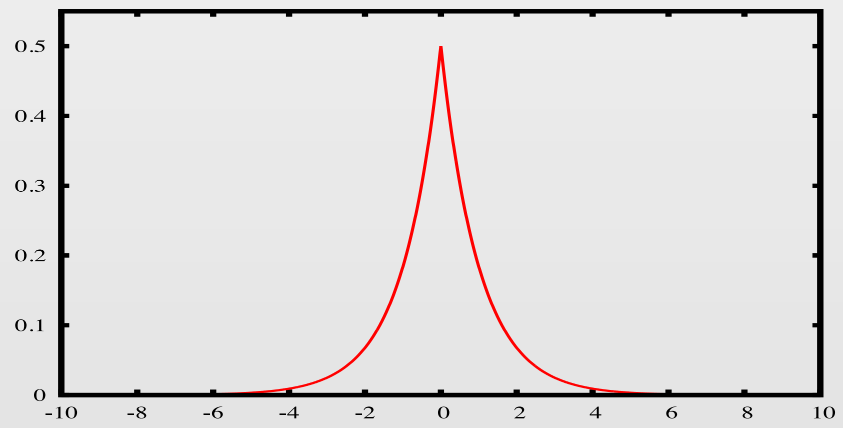
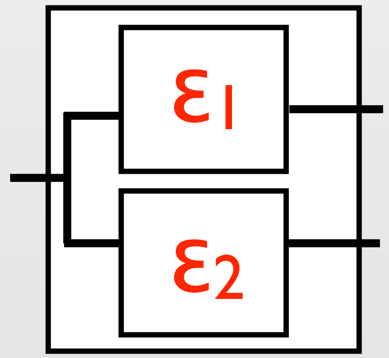
Language-based tool support available





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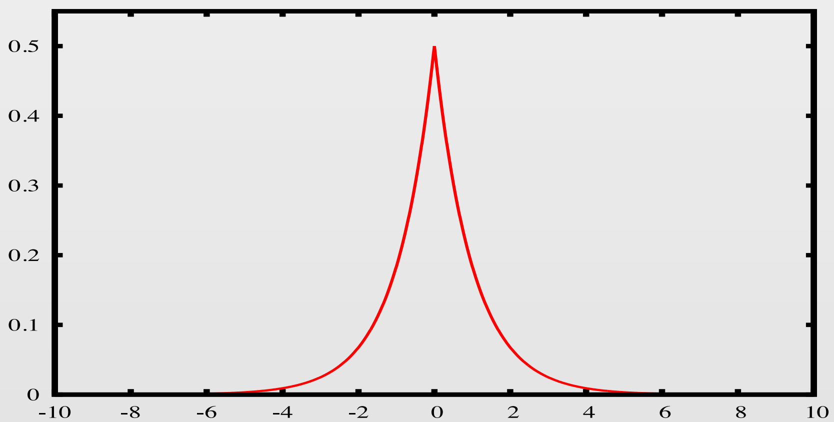

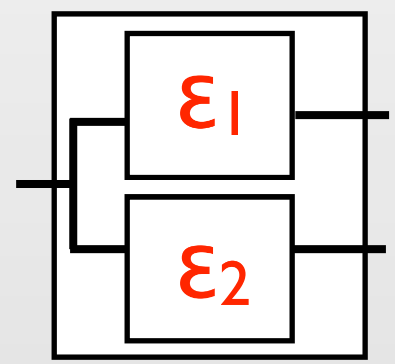

$\varepsilon_1+\varepsilon_2$

Language-based tool support available

- Expanding frontiers
  - Mechanisms: exponential, median...
  - Algorithms: streaming/graph/... algorithms
  - Definitions: approximate differential privacy, pan privacy...

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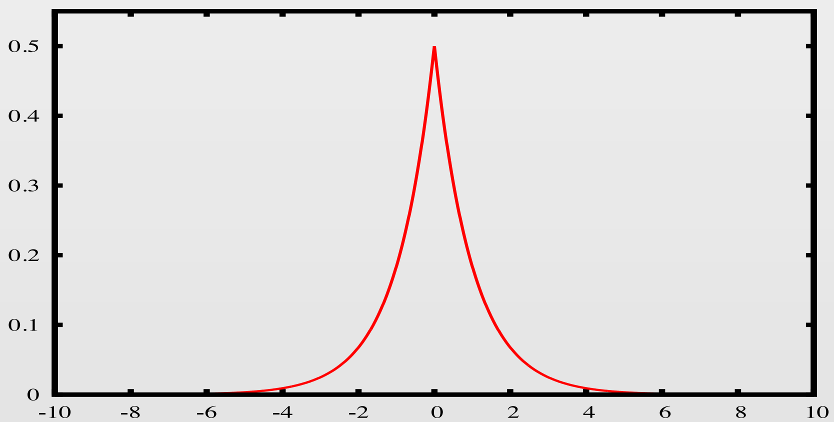

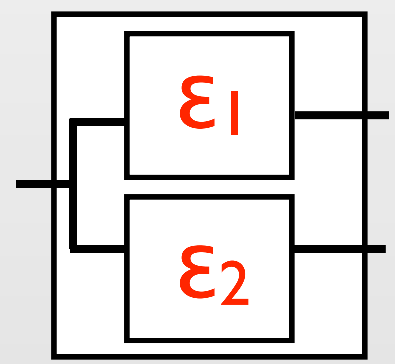

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Increasingly complex, but not supported by existing tools!

Our Contribution: CertiPriv

- Allows reasoning about approximate quantitative properties of randomized computations
  - Built from first principles and fully formalized in Coq
  - Machine-checked proofs of differential privacy
    - Correctness of Laplacian and Exponential mechanisms
    - State-of-art graph and streaming algorithms
- Generalizes CertiCrypt and opens new applications to crypto

Differential privacy as quantitative 2-safety

- K is (ε, δ)-diff. private w.r.t. Φ iff for all D_1 and D_2 and S

$$\Phi(D_1, D_2) \Longrightarrow \Pr[K(D_1) \in S] \leq \exp(\epsilon) \times \Pr[K(D_2) \in S] + \delta$$

Relational pre-condition

(Quantitative) relational post-condition

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(Quantitative) relational post-condition

- We propose a quantitative probabilistic relational Hoare Logic

$$c_1 \sim_{\alpha,\delta} c_2 : \Phi \Longrightarrow \Psi$$

such that c is (ε, δ)-diff. private w.r.t. Φ iff

$$c \sim_{\exp(\epsilon),\delta} c : \Phi \Longrightarrow \equiv$$

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Needs to be lifted to distributions

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Characterizing differential privacy

$c_1 \sim_{\alpha,\delta} c_2 : \Phi \Rightarrow \Psi$ is valid iff for all D_1 and D_2

$$\Phi(D_1, D_2) \Longrightarrow \mathrm{lift}_{\alpha,\delta}\ \Psi\ (\llbracket c_1 \rrbracket\ D_1)\ (\llbracket c_2 \rrbracket\ D_2)$$

We define α-distance such that:

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- Fundamental property of lifting

$$\Delta_\alpha(\mu_1, \mu_2) \leq \delta \iff \mathrm{lift}_{\alpha,\delta} \equiv \mu_1\ \mu_2$$

Lifting relations to distributions

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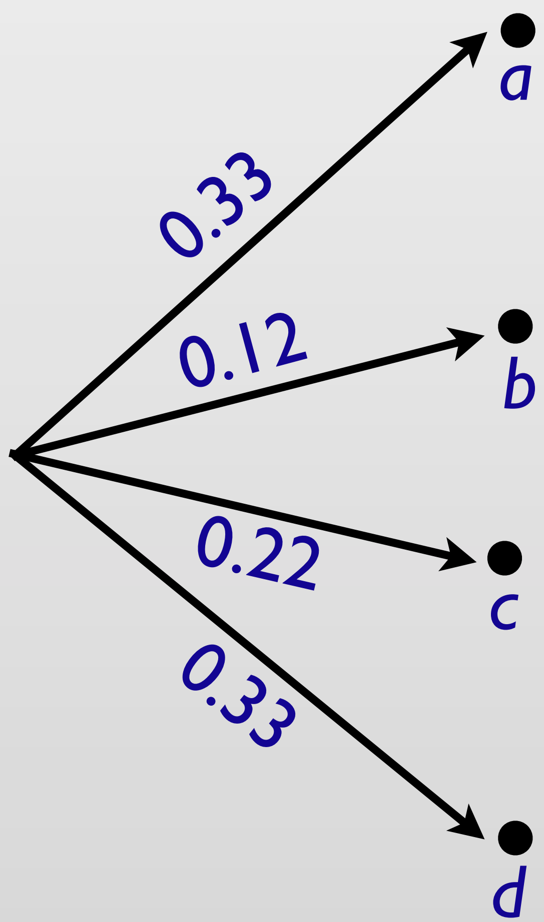

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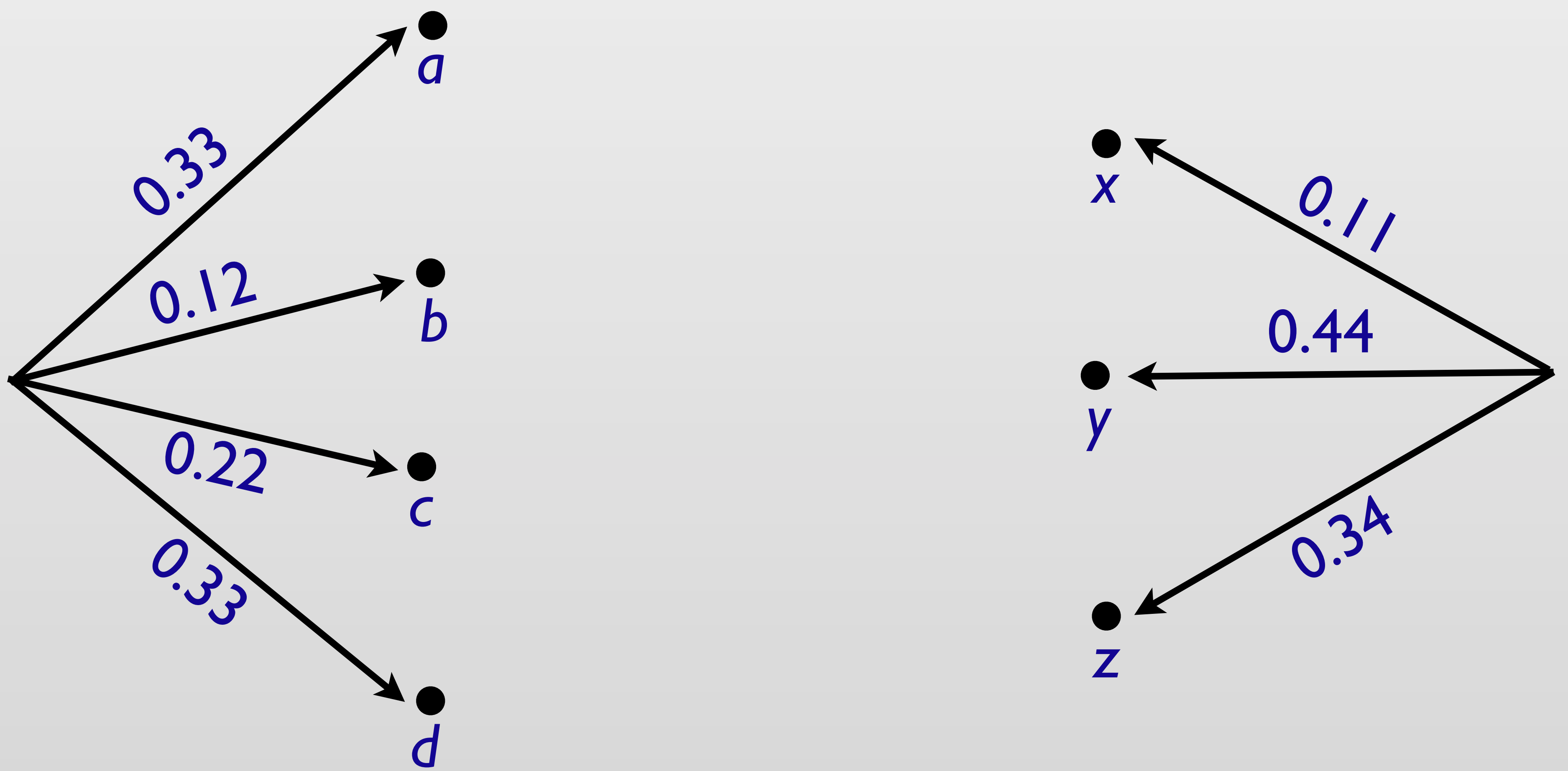
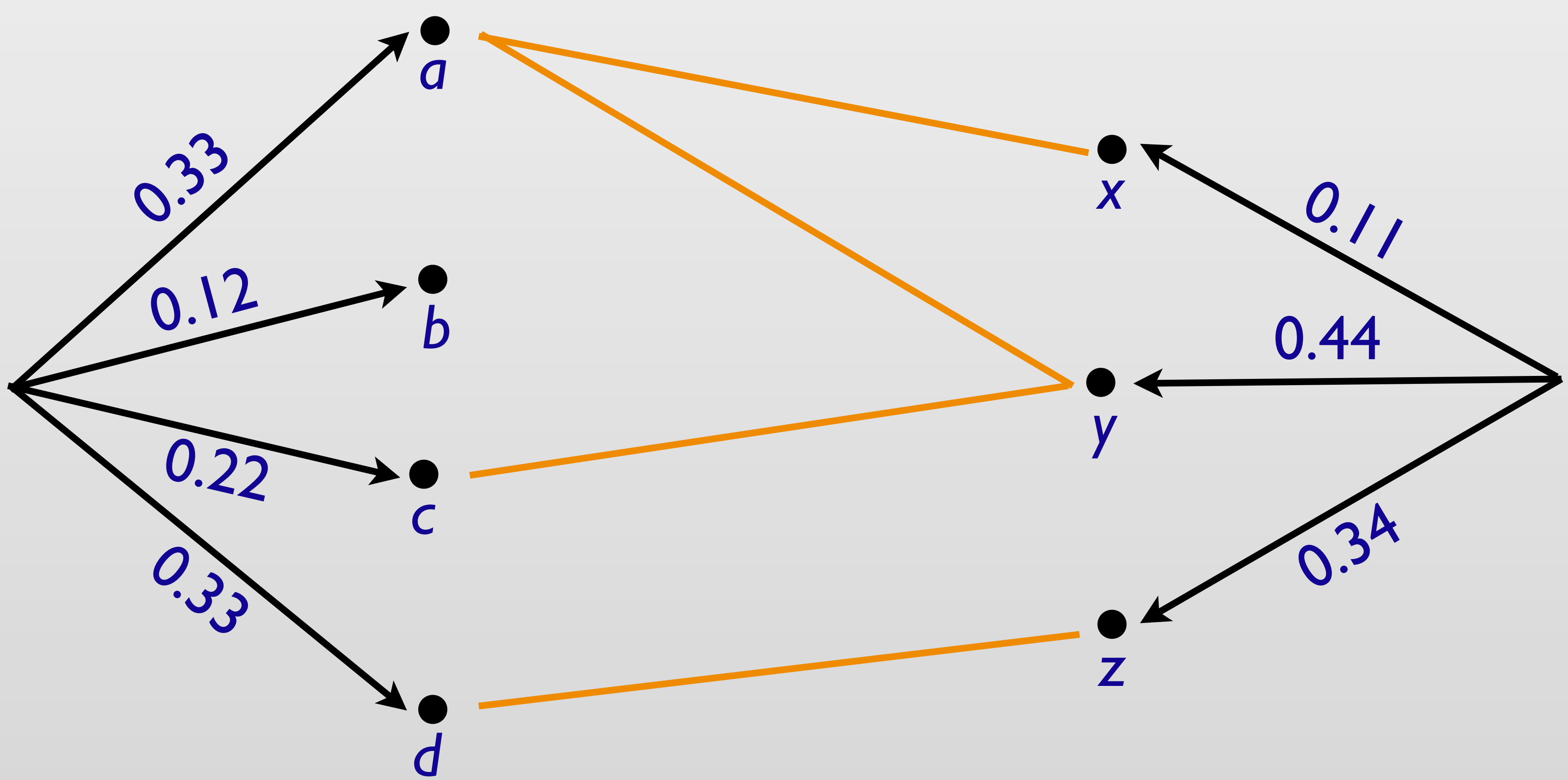


Lifting relations to distributions

Given R = {(a,x), (a,y), (c,y), (d,z)}



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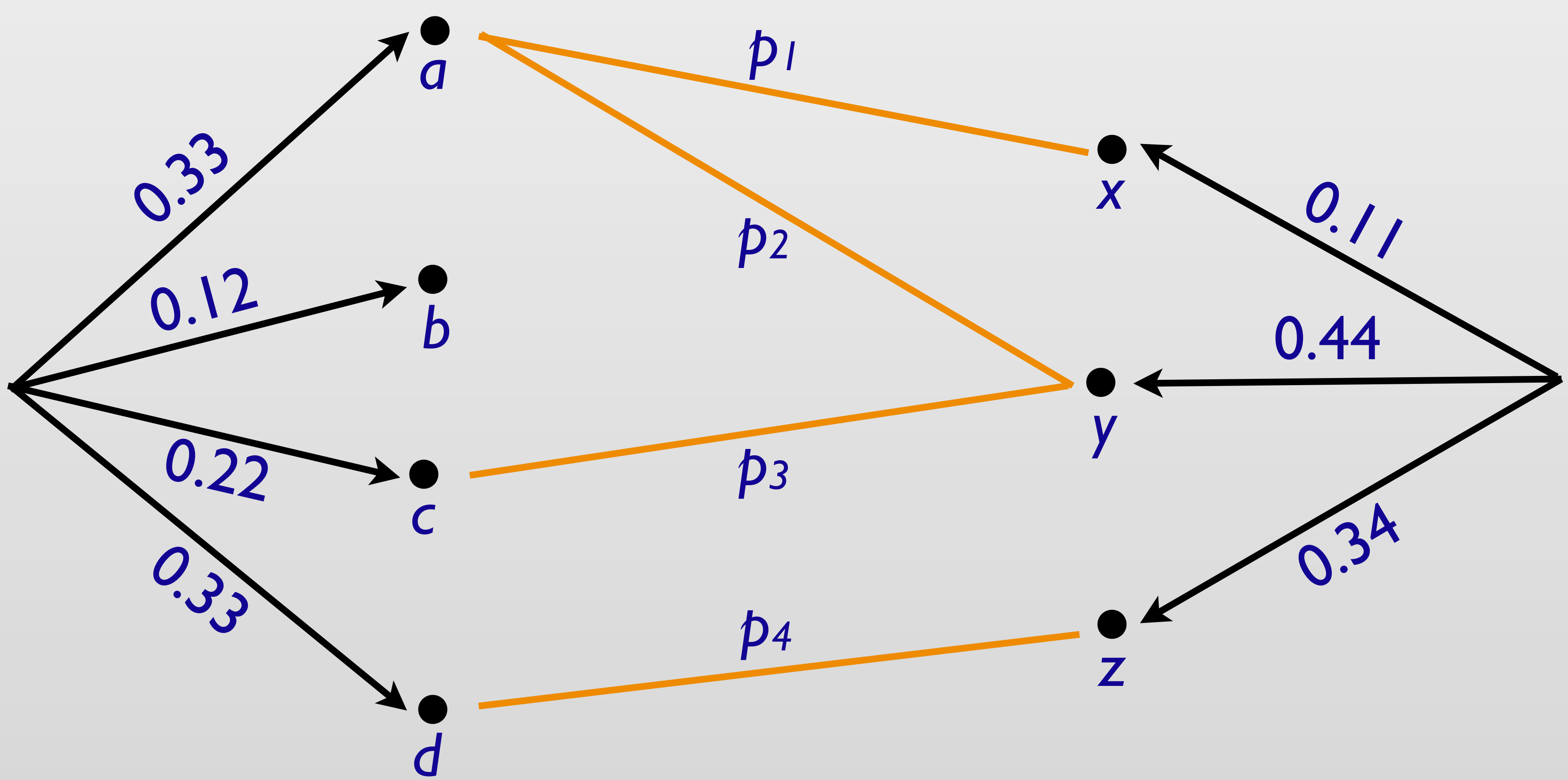
Given R = {(a,x), (a,y), (c,y), (d,z)}, α=1.1 and δ=0.01



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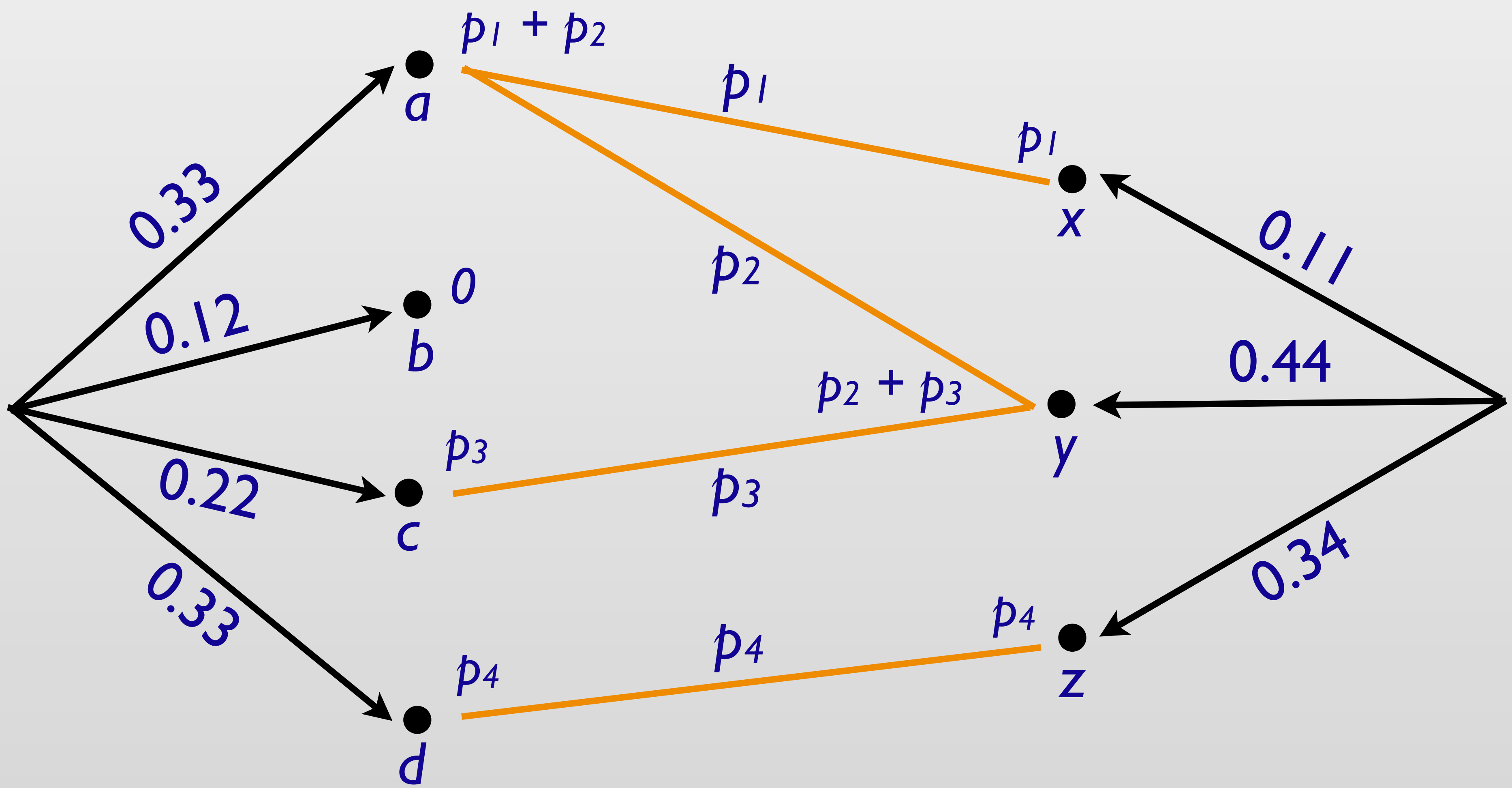
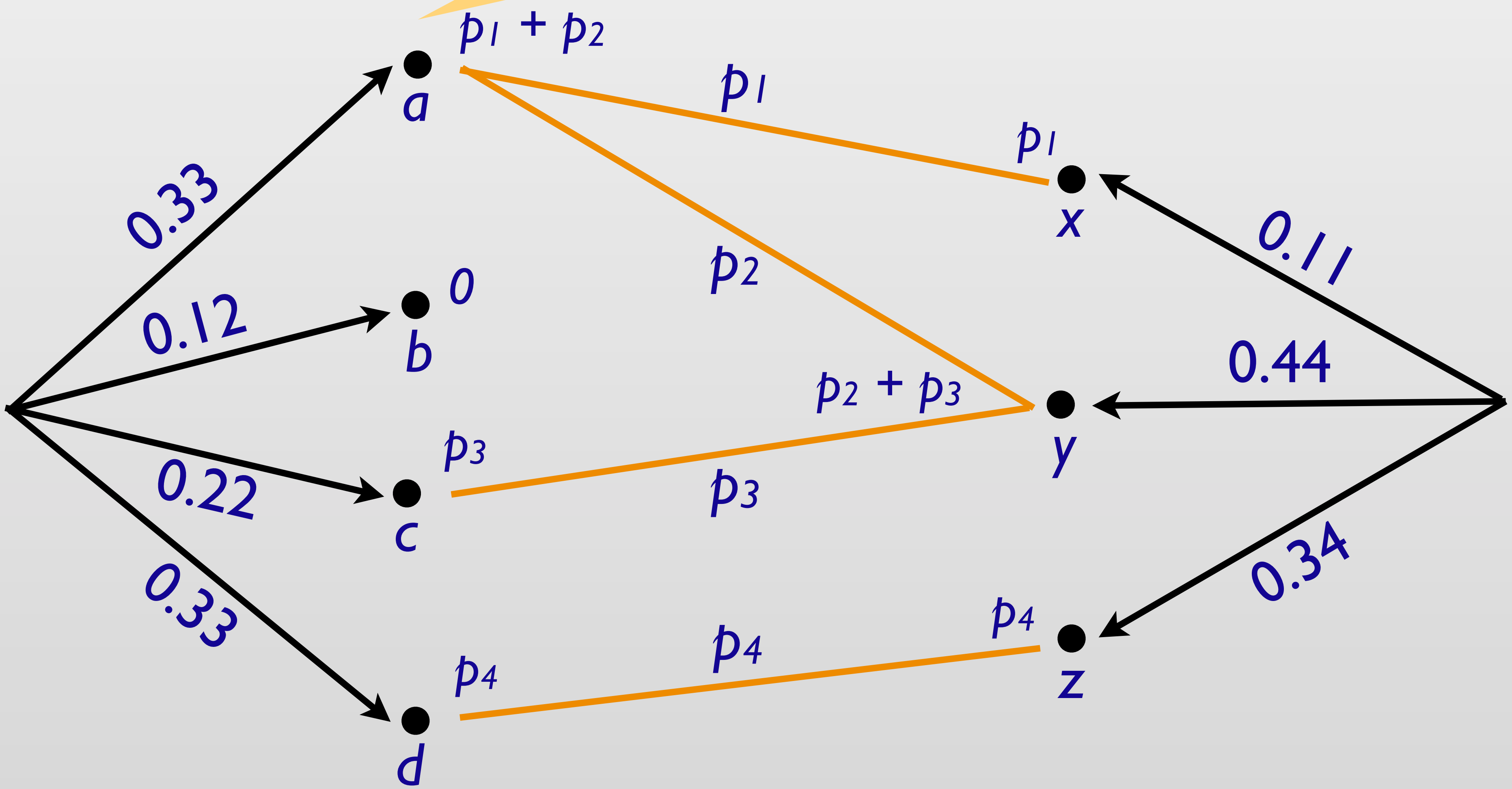


Lifting relations to distributions

Given R = {(a,x), (a,y), (c,y), (d,z)}, α=1.1 and δ=0.01

$$p_1 + p_2 \leq 0.33$$
$$\delta_a = \max\{0, 0.33 - \alpha\,(p_1 + p_2)\}$$
$$\delta_a + \delta_b + \delta_c + \delta_d \leq \delta$$



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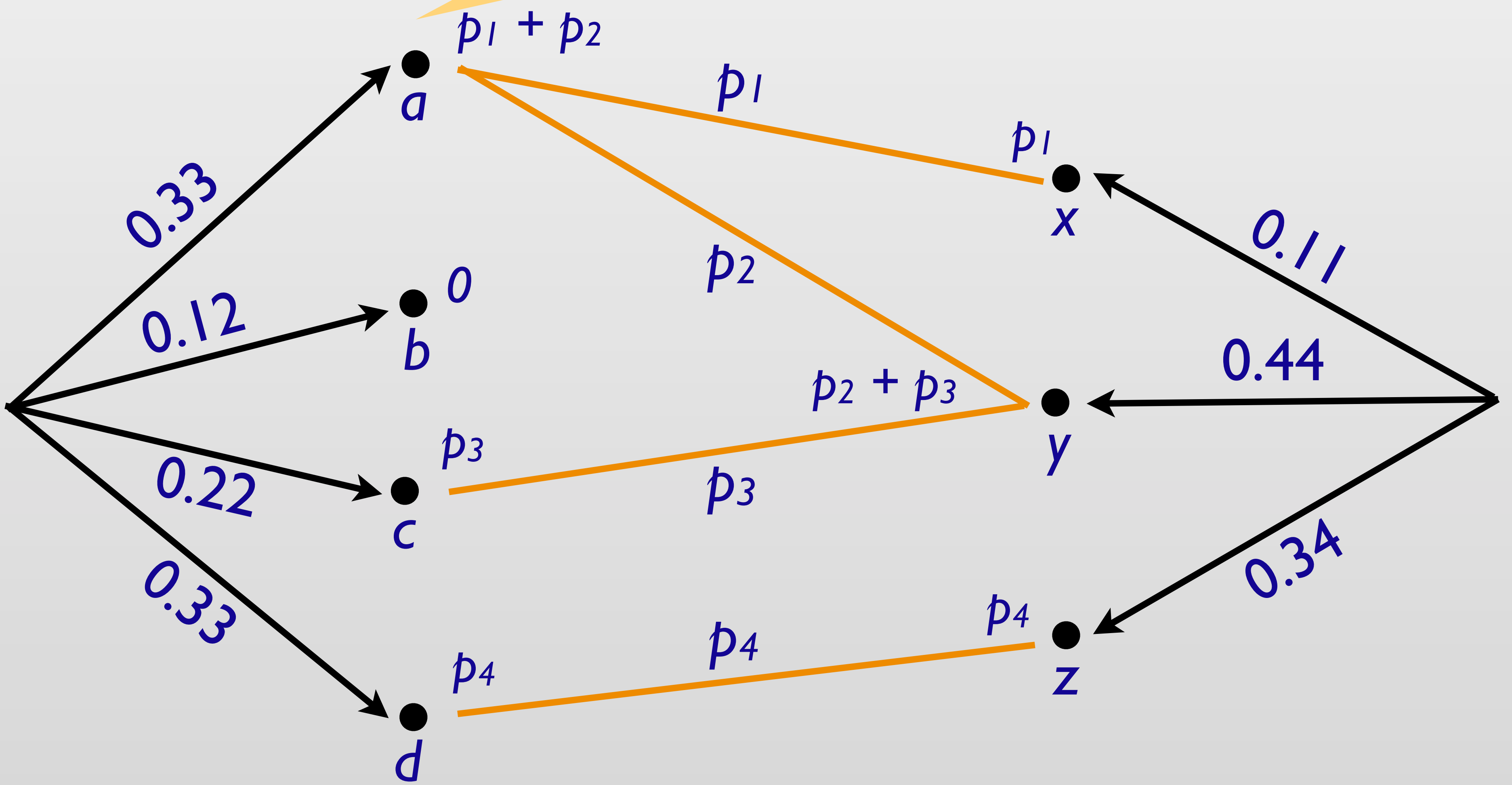


Witness distribution

$X \times Y$	$\mu(\cdot,\cdot)$
(a, x)	0.10
(a, y)	0.20
(c, y)	0.20
(d, z)	0.30
…	0

Selected rules

Sequential composition

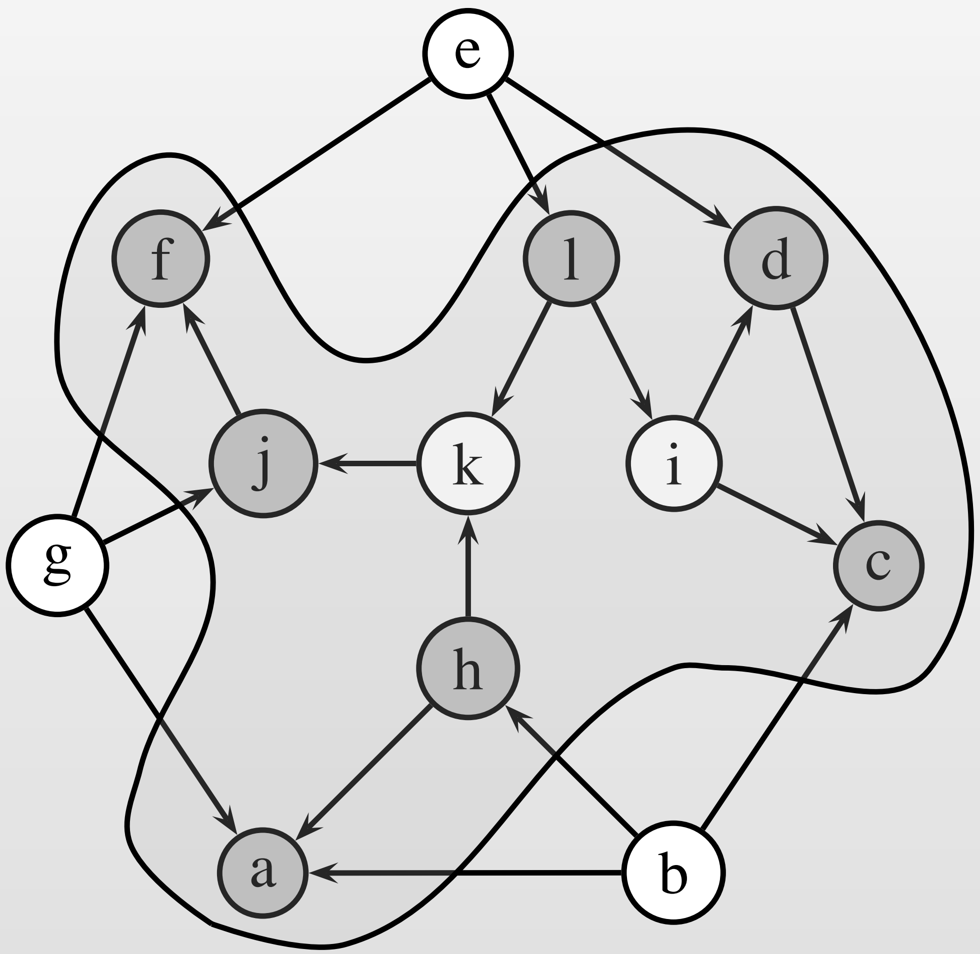
$$\frac{\models c_1 \sim_{\alpha,\delta} c_2 : \Psi \Rightarrow \Phi' \qquad \models c_1' \sim_{\alpha',\delta'} c_2' : \Phi' \Rightarrow \Phi}{\models c_1; c_1' \sim_{\alpha\alpha',\delta+\delta'} c_2; c_2' : \Psi \Rightarrow \Phi}$$

Laplacian Mechanism

$$\frac{}{\models x \xleftarrow{\$} \mathcal{L}_\lambda(r) \sim_{\exp(\epsilon),0} y \xleftarrow{\$} \mathcal{L}_\lambda(s) : |r\langle 1\rangle - s\langle 2\rangle| \leq \lambda\epsilon \Rightarrow x\langle 1\rangle = y\langle 2\rangle}$$

Application: Vertex Cover

Gupta et al. [SODA '10]



VertexCover(V, E)

$\pi \leftarrow \text{nil};$

while $E \neq \emptyset$ do

$\quad v \xleftarrow{\$} \text{pick}(V, E);$

$\quad \pi \leftarrow v :: \pi;$

$\quad V \leftarrow V \setminus \{v\};\ \ E \leftarrow E \setminus (\{v\} \times V);$

end

$$\text{pick}(V, E) \;\propto\; deg_E(v)$$

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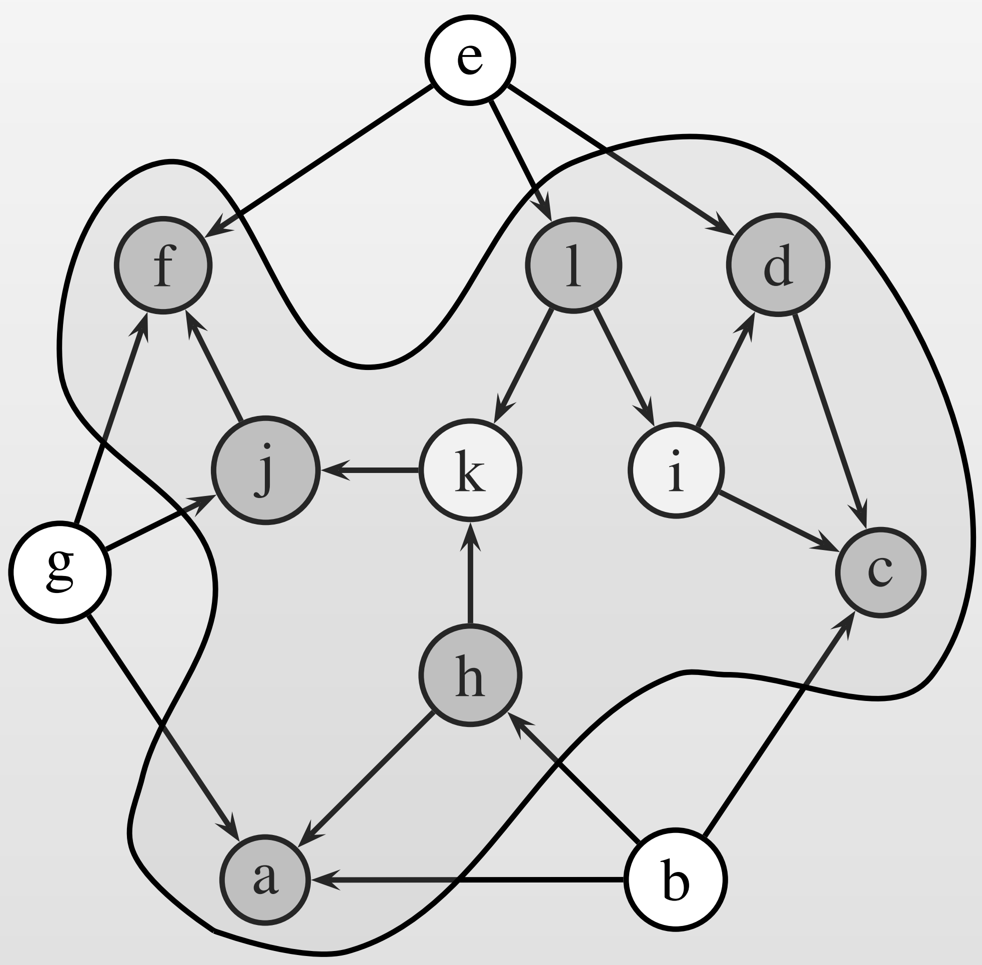

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$$\text{VertexCover}(V, E) \sim_{\exp(\epsilon),0} \text{VertexCover}(V, E) :$$

$$V\langle 1\rangle = V\langle 2\rangle \wedge E\langle 1\rangle = E\langle 2\rangle \cup \{(t, u)\} \implies \pi\langle 1\rangle = \pi\langle 2\rangle$$

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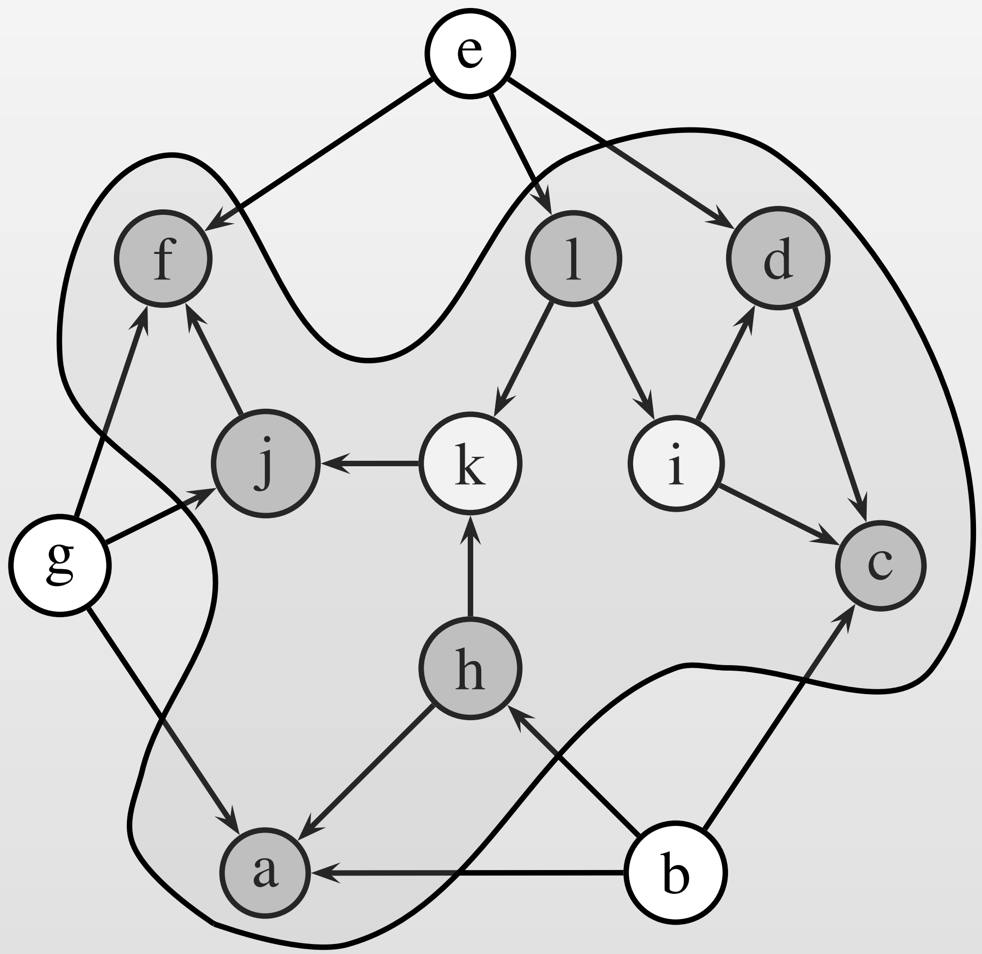

$\mathsf{VertexCover}(V, E)$

```
π ← nil;
while E ≠ ∅ do
    v ←$ pick(V, E);
    π ← v :: π;
    V ← V\{v};  E ← E \ ({v} × V);
end
```

$\mathsf{pick}(V, E) \;\propto\; deg_E(v)$

Not satisfied!

$$\mathsf{VertexCover}(V, E) \sim_{\exp(\epsilon),0} \mathsf{VertexCover}(V, E) :$$

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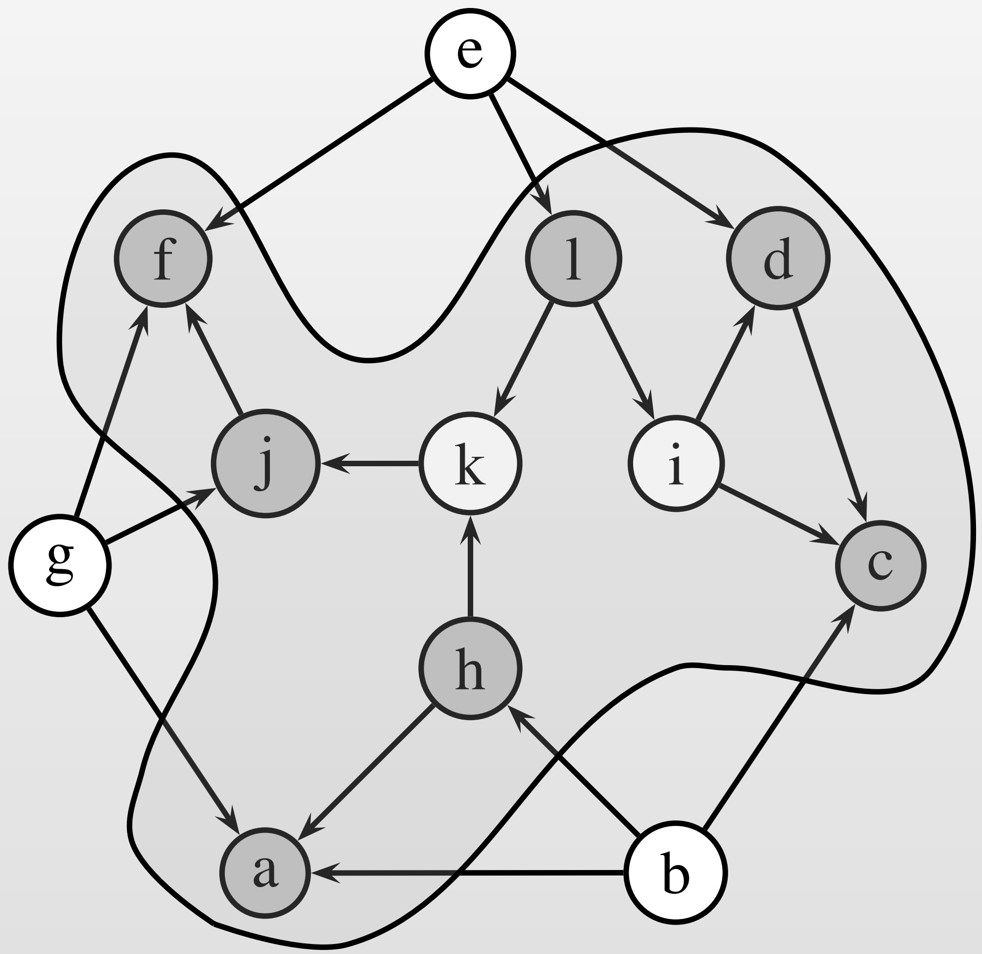

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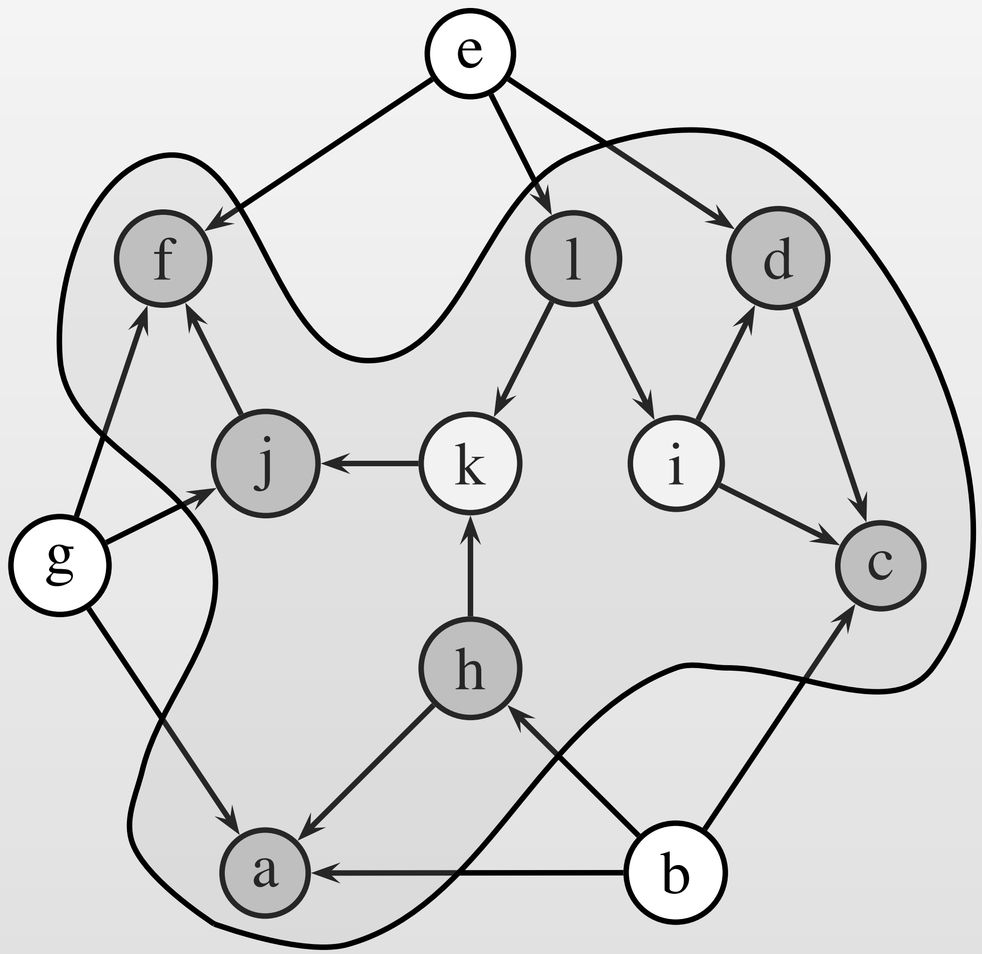

$\pi = [b, g, e, h, l, k,$
$j, i, f, d, c, a]$

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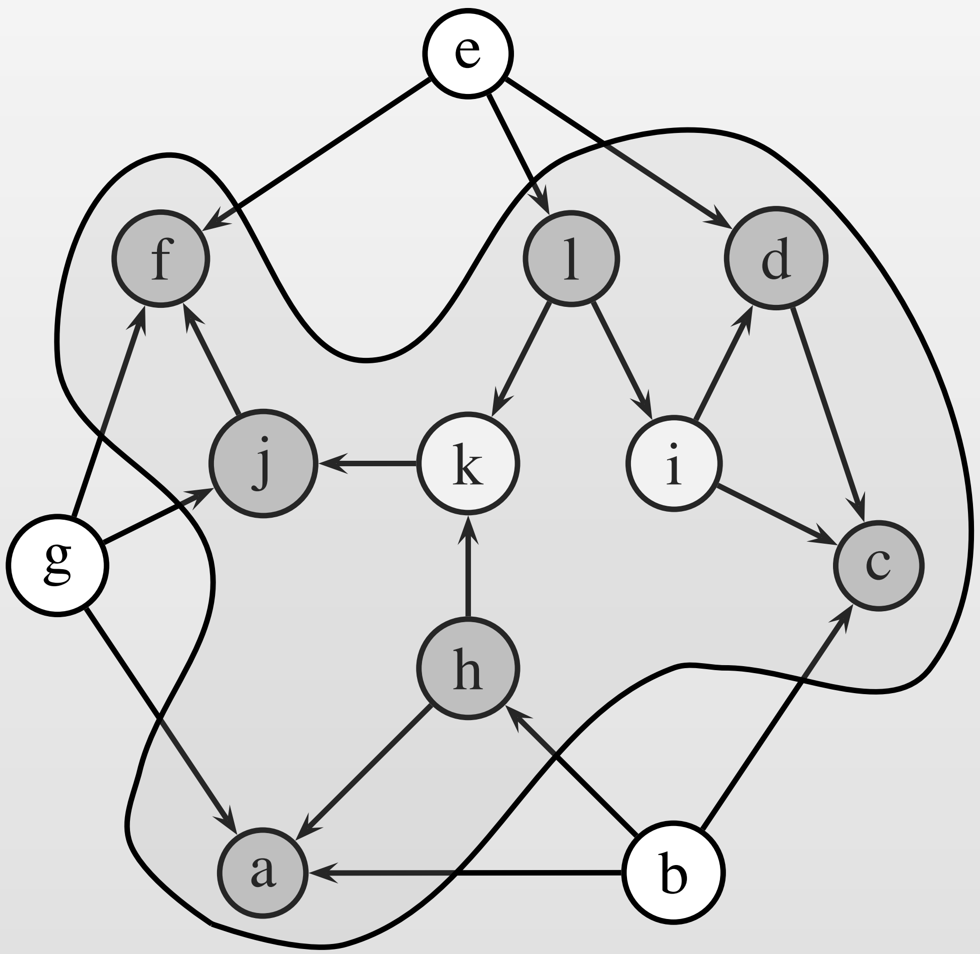

$\pi = [b, g, e, h, l, k, j, i, f, d, c, a]$

VertexCover(V, E, ϵ)

$\pi \leftarrow$ nil; $n \leftarrow |V|$; $i \leftarrow 0$;

while $i < n$ do

  $v \xleftarrow{\$}$ pick(V, E, ϵ, n, i);

  $\pi \leftarrow v :: \pi$;

  $V \leftarrow V \setminus \{v\}$; $E \leftarrow E \setminus (\{v\} \times V)$;

  $i \leftarrow i + 1$

end

$$\text{pick}(V, E, \epsilon, n, i) \;\propto\; deg_E(v) + \frac{4}{\epsilon}\sqrt{\frac{n}{n-i}}$$

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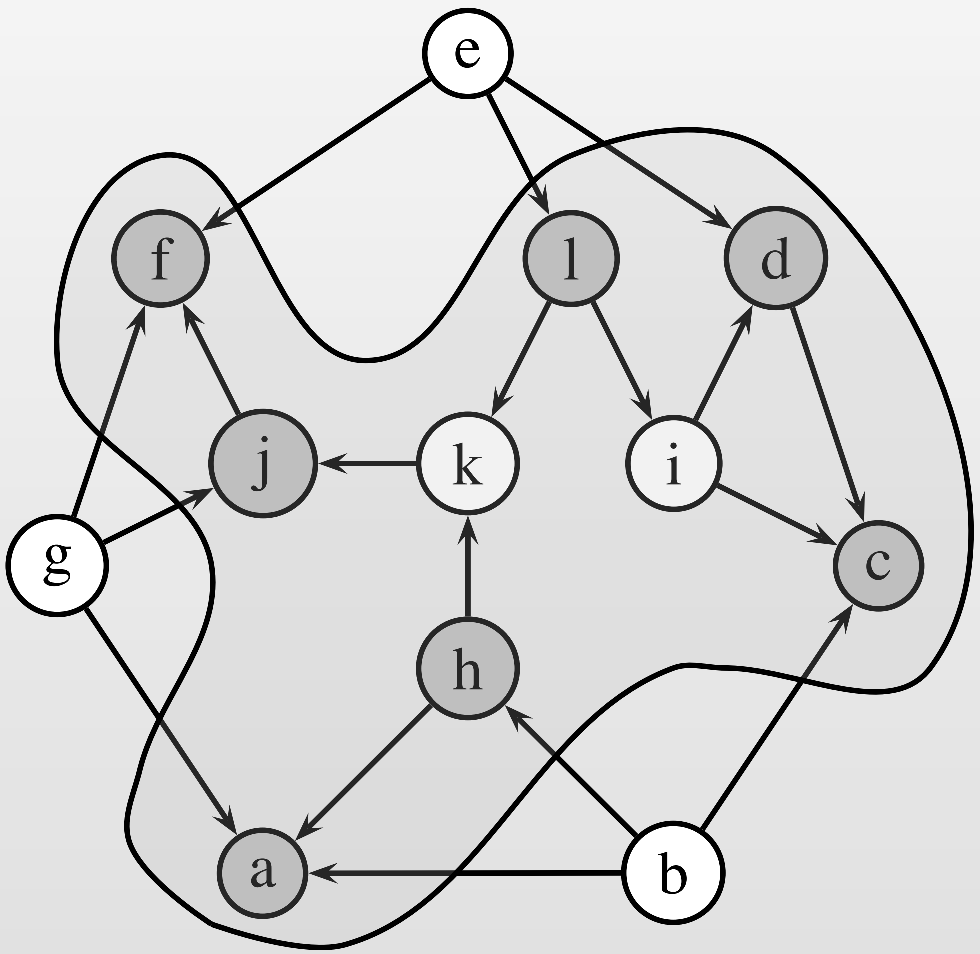

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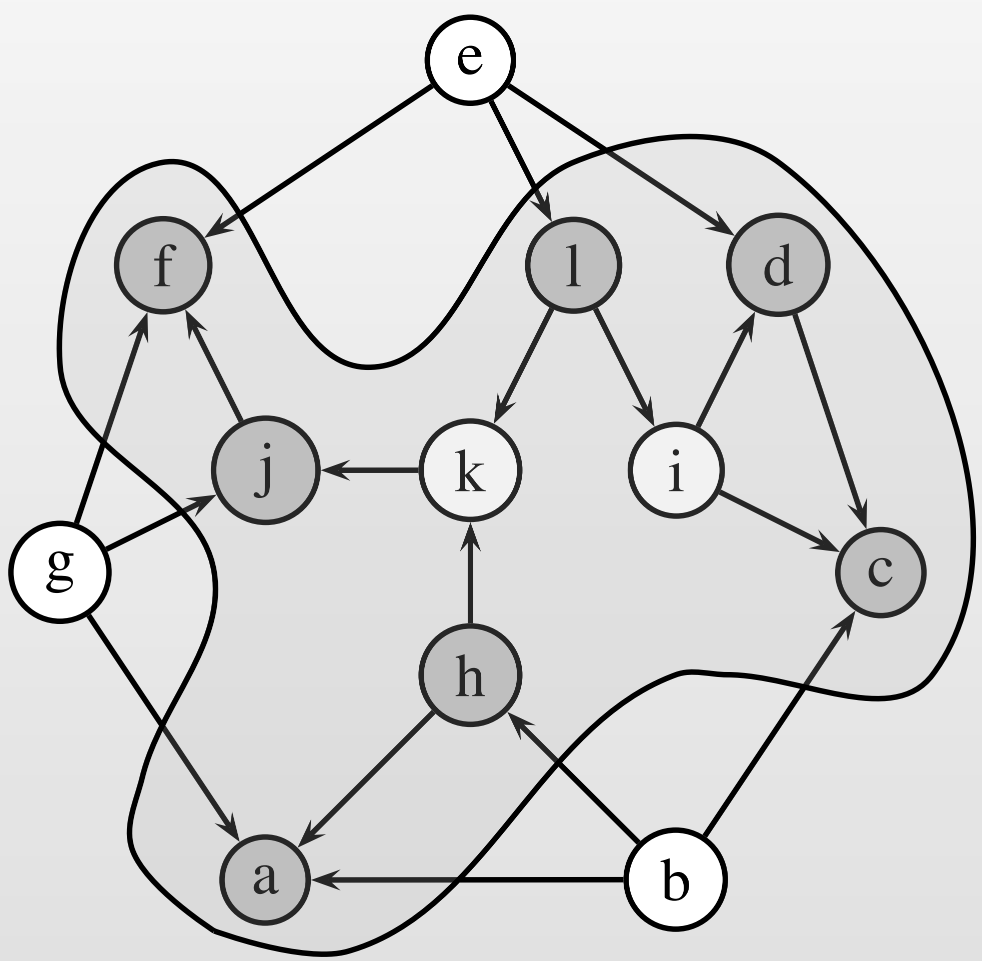

$\mathsf{VertexCover}(V, E, \epsilon)$

$$
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\mathsf{end}
\end{array}
$$

$\mathsf{pick}(V, E, \epsilon, n, i) \ \propto \ a$

Proven correct using CertiPriv

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$$V\langle 1\rangle = V\langle 2\rangle \wedge E\langle 1\rangle = E\langle 2\rangle \cup \{(t, u)\} \implies \pi\langle 1\rangle = \pi\langle 2\rangle$$

Conclusions

- Framework for reasoning about quantitative relational properties of randomized computations
  - Laplacian and Exponential mechanisms
  - Differential privacy for streaming and graph algorithms
  - Asymmetric logic
- Further work:
  - Computational differential privacy
  - Hash functions unto elliptic curves and statistical zero-knowledge
- Challenge: logic for arbitrary quantitative relational properties

Thanks for your attention!

Define α-distance as:

$$\Delta_\alpha(d_1, d_2) = \max_A(\max(d_1\ 1_A - \alpha\ (d_2\ 1_A), d_2\ 1_A - \alpha\ (d_1\ 1_A)))$$

(α,δ)-lifting of relations to distributions:

$$\begin{aligned}\text{lift}_{\alpha,\delta}\ R\ (d_1 : \mathcal{D}_A)\ (d_2 : \mathcal{D}_B) = \exists (d : \mathcal{D}_{A*B}),\\ \pi_1(d) \leq d_1 \wedge \Delta_\alpha(\pi_1(d), d_1) \leq \delta \wedge\\ \pi_2(d) \leq d_2 \wedge \Delta_\alpha(\pi_2(d), d_2) \leq \delta \wedge\ \text{range}\ R\ d\end{aligned}$$

Output perturbation makes numerical queries ε-diff. private

- The Φ-sensitivity of a query $f : \mathcal{D} \to \mathbb{R}$ is defined as:

$$\Delta(f) = \max\{f(D_1) - f(D_2) \mid \Phi(D_1, D_2)\}$$

- The randomized computation

$$K(D) = f(D) + \mathsf{Lap}(\Delta(f)/\epsilon)$$

is ε-differentially private

Density proportional to $\exp(-\epsilon/\Delta(f))$

Composition theorems

If K_1 is $(\varepsilon_1, \delta_1)$-diff. private and K_2 is $(\varepsilon_2, \delta_2)$-diff. private

- Sequential composition

$(\varepsilon_1+\varepsilon_2, \delta_1+\delta_2)$-diff. private

- Parallel composition

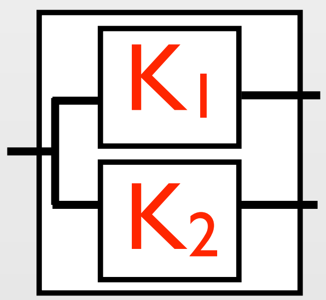
K_1 and K_2 depend on disjoint parts of the database

$(\max\{\varepsilon_1, \varepsilon_2\}, \max\{\delta_1, \delta_2\})$-diff. private

Composition theorems

If K_1 is $(\varepsilon_1, \delta_1)$-diff. private and K_2 is $(\varepsilon_2, \delta_2)$-diff. private

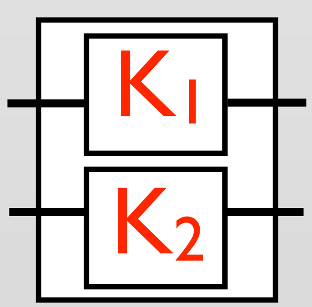
- Sequential composition



$(\varepsilon_1+\varepsilon_2, \delta_1+\delta_2)$-diff. private

- Parallel composition

K_1 and K_2 depend on disjoint parts of the database



$(\max\{\varepsilon_1, \varepsilon_2\}, \max\{\delta_1,\delta_2\})$-diff. private