Equations in Free Semigroups with Anti-involution and their Relation to Equations in Free Groups

Claudio Gutiérrez[1,2]

[1] Computer Science Group, Dept. of Mathematics, Wesleyan University.
[2] Departamento de Ingeniería Matemática, D.I.M., Universidad de Chile.
(Research funded by FONDAP, Matemáticas Aplicadas.)
cgutierrez@wesleyan.edu

Abstract. The main result of the paper is the reduction of the problem of satisfiability of equations in free groups to the satisfiability of equations in free semigroups with anti-involution (SGA), by a non-deterministic polynomial time transformation.
A free SGA is essentially the set of words over a given alphabet plus an operator which reverses words. We study equations in free SGA, generalizing several results known for equations in free semigroups, among them that the exponent of periodicity of a minimal solution of an equation E in free SGA is bounded by $2^{\mathcal{O}(|E|)}$.

1 Introduction

The study of the problem of solving equations in free SGA (unification in free SGA) and its computational complexity is a problem closely related to the problem of solving equations in free semigroups and in free groups, which lately have attracted much attention of the theoretical computer science community [3], [12], [13], [14].

Free semigroups with anti-involution is a structure which lies in between that of free semigroups and free groups. Besides the relationship with semigroups and groups, the axioms defining SGA show up in several important theories, like algebras of binary relations, transpose in matrices, inverse semigroups.

The problem of solving equations in free semigroups was proven to be decidable by Makanin in 1976 in a long paper [10] . Some years later, in 1982, again Makanin proved that solving equations in free groups was a decidable problem [11]. The technique used was similar to that of the first paper, although the details are much more involved. He reduced equations in free groups to solving equations in free SGA with special properties ('non contractible'), and showed decidability for equation of this type. For free SGA (without any further condition) the decidability of the problem of satisfiability of equations is still open, although we conjecture it is decidable.

Both of Makanin's algorithms have received very much attention. The enumeration of all unifiers was done by Jaffar for semigroups [6] and by Razborov for groups [15]. Then, the complexity has become the main issue. Several authors have analyzed the complexity of Makanin's algorithm for semigroups [6], [16], [1], being EXPSPACE the best upper-bound so far [3]. Very recently Plandowski, without using Makanin's algorithm, presented an upper-bound of PSPACE for the problem of satisfiability of equations in free semigroups [14]. On the other hand, the analysis of the complexity of Makanin's algorithm for groups was done by Koscielski and Pacholski [8], who showed that it is not primitive recursive.

With respect to lower bounds, the only known lower bound for both problems is NP-hard, which seems to be weak for the case of free groups. It is easy to see that this lower bound works for the case of free SGA as well.

The main result of this paper is the reduction of equations in free groups to equations in free SGA (Theorem 9 and Corollary 10). This is achieved by generalizing to SGA several known results for semigroups, using some of Makanin's results in [11], and proving a result that links these results (Proposition 3). Although we do not use it here, we show that the standard bounds on the exponent of periodicity of minimal solutions to word equations also hold with minor modifications in the case of free SGA (Theorem 5).

For concepts of word combinatorics we will follow the notation of [9]. By ϵ we denote the empty word.

2 Equations in Free SGA

A *semigroup with anti-involution* (SGA) is an algebra with a binary associative operation (written as concatenation) and a unary operation $(\ )^{-1}$ with the equational axioms

$$(xy)z = x(yz), \qquad (xy)^{-1} = y^{-1}x^{-1}, \qquad x^{-1-1} = x. \tag{1}$$

A *free* semigroup with anti-involution is an initial algebra for this variety. It is not difficult to check that for a given alphabet C, the set of words over $C \cup C^{-1}$ together with the operator $(\ )^{-1}$, which reverses a word and changes every letter to its twin (e.g. a to a^{-1} and conversely) is a free algebra for SGA over A.

Equations and Solutions. Let C and V be two disjoint alphabets of constants and variables respectively. Denote by $C^{-1} = \{c^{-1} : c \in C\}$. Similarly for V^{-1}. An *equation* E in free SGA with constants C and variables V is a pair (w_1, w_2) of words over the alphabet $\mathcal{A} = C \cup C^{-1} \cup V \cup V^{-1}$. The number $|E| = |w_1| + |w_2|$ is the *length* of the equation E and $|E|_V$ will denote the number of occurrences of variables in E. These equations are also known as *equations in a paired alphabet.*

A map $S : V \longrightarrow (C \cup C^{-1})^*$ can be uniquely extended to a SGA-homomorphism $\bar{S} : \mathcal{A}^* \longrightarrow (C \cup C^{-1})^*$ by defining $S(c) = c$ for $c \in C$ and $S(u^{-1}) = (S(u))^{-1}$ for $u \in C \cup V$. We will use the same symbol S for the map S and the SGA-homomorphism $\bar{S}$. A *solution* S of the equation $E = (w_1, w_2)$ is (the unique SGA-homomorphism defined by) a map $S : V \longrightarrow (C \cup C^{-1})^*$ such that $S(w_1) = S(w_2)$. The length of the solution S is $|S(w_1)|$. By $S(E)$ we denote the word $S(w_1)$ (which is the same as $S(w_2)$). Each occurrence of a symbol

$u \in \mathcal{A}$ in E with $S(u) \neq \epsilon$ determines a unique factor in $S(E)$, say $S(E)[i,j]$, which we will denote by $S(u,i,j)$ and call simply an *image* of u in $S(E)$.

The Equivalence Relation (S,E). Let S be a solution of E and P be the set of positions of $S(E)$. Define the binary relation $(S,E)'$ in $P \times P$ as follows: given positions $p,q \in P$, $p(S,E)'q$ if and only if one of the following hold:

1. $p = i + k$ and $q = i' + k$, where $S(x,i,j)$ and $S(x,i',j')$ are images of x in $S(E)$ and $0 \leq k < |S(x)|$.
2. $p = i + k$ and $q = j' - k$, where $S(x,i,j)$ and $S(x^{-1},i',j')$ are images of x and x^{-1} in $S(E)$ and $0 \leq k < |S(x)|$.

Then define (S,E) as the transitive closure of $(S,E)'$. Observe that (S,E) is an equivalence relation.

Contractible Words. A word $w \in \mathcal{A}^*$ is called *non-contractible* if for every $u \in \mathcal{A}$ the word w contains neither the factor uu^{-1} nor $u^{-1}u$. An equation (w_1,w_2) is called non-contractible if both w_1 and w_2 are non-contractible. A solution S to an equation E is called non-contractible if for every variable x which occurs in E, the word $S(x)$ is non-contractible.

Boundaries and Superpositions. Given a word $w \in \mathcal{A}^*$, we define a *boundary* of w as a pair of consecutive positions $(p, p+1)$ in w. We will write simply p_w, the subindex denoting the corresponding word. By extension, we define $i(w) = 0_w$ and $f(w) = |w|_w$, the *initial* and *final* boundaries respectively. Note that the boundaries of w have a natural linear order ($p_w \leq q_w$ iff $p \leq q$ as integers).

Given an equation $E = (w_1,w_2)$, a *superposition* (of the boundaries of the left and right hand sides) of E is a linear order $\leq$ of the set of boundaries of w_1 and w_2 extending the natural orders of the boundaries of w_1 and w_2, such that $i(w_1) = i(w_2)$ and $f(w_1) = f(w_2)$ and possibly identifying some p_{w_1} and q_{w_2}.

Cuts and Witnesses. Given a superposition $\leq$ of $E = (w_1,w_2)$, a *cut* is a boundary j of w_2 (resp. w_1) such that $j \neq b$ for all boundaries b of w_1 (resp. w_2). Hence a cut determines at least three symbols of E, namely $w_2[j]$, $w_2[j+1]$ and $w_1[i+1]$, where i is such that $i_{w_1} < j_{w_2} < (i+1)_{w_1}$ in the linear order, see Figure 1. The triple of symbols $(w_2[j], w_2[j+1], w_1[i])$ is called a *witness* of the cut. A superposition is called *consistent* if $w_1[i+1]$ is a variable.

Observe that every superposition gives rise to a system of equations $(E, \leq)$, which codifies the constraints given by $\leq$, by adding the corresponding equations and variables $x = x'y$ which the cuts determine. Also observe that every solution S of E determines a unique consistent superposition, denoted $\leq_S$. Note finally that the cut j determines a boundary $(r, r+1)$ in $S(E)$; if $p \leq r < q$, we say that the subword $S(E)[p,q]$ of $S(E)$ *contains* the cut j.

Lemma 1 *Let E be an equation in free SGA. Then E has a solution if and only if $(E,\leq)$ has a solution for some consistent superposition $\leq$. There are no more than $|E|^{4|E|_V}$ consistent superpositions.*

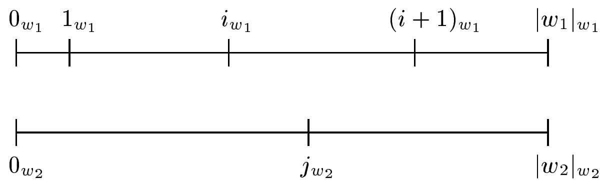

Fig. 1. The cut j_w.

Proof. Obviously if for some consistent superposition $\leq$, $(E, \leq)$ has a solution, then E has a solution. Conversely, if E has a solution S, consider the superposition generated by S.

As for the bound, let $E = (w_1, w_2)$ and write v for $|E|_V$. First observe that if w_2 consists only of constants, then there are at most $|w_2|^v$ consistent superpositions. To get a consistent superposition in the general case, first insert each initial and final boundary of each variable in w_2 in the linear order of the boundaries of w_1 (this can be done in at most $|E| + v$ ways). Then it rest to deal with the subwords of w_2 in between variables (hence consisting only of constants and of total length $\leq |E| - v$). Summing up, there are no more than $(|E| + v)^{2v}(|E| - v)^v \leq |E|^{4v}$ consistent superpositions.

Lemma 2 (Compare Lemma 6, [12]) *Assume S is a minimal (w.r.t. length) solution of E. Then*

1. *For each subword $w = S(E)[i, j]$ with $|w| > 1$, there is an occurrence of w or w^{-1} which contains a cut of $(E, \leq_S)$.*
2. *For each letter $c = S(E)[i]$ of $S(E)$, there is an occurrence of c or c^{-1} in E.*

Proof. Let $1 \leq p \leq q \leq |S(E)|$. Suppose neither $w = S(E)[p, q]$ nor w^{-1} have occurrences in $S(E)$ which contain cuts. Consider the position p in $S(E)$ and its (S, E)-equivalence class P, and define for each variable x occurring in E,

> $S'(x)$ = the subsequence of some image $S(x, i, j)$ of x consisting of all positions which are not in the set P. (i.e. "cut off" from $S(x, i, j)$ all the positions in P).

It is not difficult to see that S' is well defined, *i.e.*, it does not depend on the particular image $S(x, i, j)$ of x chosen, and that $S'(w_1) = S'(w_2)$ (these facts follow from the definition of (S, E)-equivalence). Now, if P does not contain any images of constants of E, it is easy to see that S' is a solution of the equation E. But $|S'(E)| < |S(E)|$, which is impossible because S was assumed to be minimal.

Hence, for each word $w = S[p, q]$, its first position must in the same (S, E)-class of the position of the image of a constant c of E. If $p < q$ the right (resp. left) boundary of that constant is a cut in w (resp. w^{-1}) which is neither initial nor final (check definition of (S, E)-equivalence for $S(E)[p+1]$, etc.), and we are in case 1. If $p = q$ we are in case 2.

Proposition 3 *For each non-contractible equation E there is a finite list of systems of equations $\Sigma_1, \ldots, \Sigma_k$ such that the following conditions hold:*

1. *E has a non-contractible solution if and only if one Σ_i has a solution.*
2. *$k \leq |E|^{8|E|_V}$.*
3. *There is $c > 0$ constant such that $|\Sigma_i| \leq c|E|$ and $|\Sigma_i|_V \leq c|E|_V$ for each $i = 1, \ldots, k$.*

Proof. Let $\leq$ be a consistent superposition of E, and let

$$(x_1, y_1, z_1), \ldots, (x_r, y_r, z_r) \tag{2}$$

be a list of those witnesses of the cuts of $(E, \leq)$ for which at least one of the x_i, y_i is a variable. Let

$$D = \{(c, d) \in (C \cup C^{-1})^2 : c \neq d^{-1} \wedge d \neq c^{-1}\},$$

and define for each r-tuple $\langle (c_i, d_i) \rangle_i$, of pairs of D the system

$$\Sigma_{\langle (c_i, d_i) \rangle_i} = (E, \leq) \cup \{(x_i, x_i' c_i), (y_i, d_i y_i') : i = 1, \ldots, r\}.$$

Now, if S is a non-contractible solution of $(E, \leq)$ then S define a solution of some Σ_i, namely the one defined by the r-tuple defined by the elements $(c_i, d_i) = (S(x_i)[|S(x_i)|], S(y_i)[1])$, for $i = 1, \ldots, r$. Note that because E and S are non-contractible, each (c_i, d_i) is in D.

On the other direction, suppose that S is a solution of some Σ_i. Then obviously S is a solution of $(E, \leq)$. We only need to prove that the $S(z)$ is non-contractible for all variables z occurring in E. Suppose some z has a factor cc^{-1}, for $c \in C$. Then by Lemma 2 there is an occurrence of cc^{-1} (its converse is the same) which contains a cut of $(E, \leq)$. But because E is non-contractible, we must have that one of the terms in (2), say (x_j, y_j, z_j), witnesses this occurrence, hence $x_j = x_j' c$ and $y_j = c^{-1} y_j'$, which is impossible by the definition of the Σ_i's.

The bound in 2. follows by simple counting: observe that $r \leq 2|E|_V$ and $|D| \leq |C|^{2r} \leq |E|^{4|E|_V}$, and the number k of systems is no bigger than the number of superpositions times $|D|$. For the bounds in 3. just sum the corresponding numbers of the new equations added.

The following is an old observation of Hmelevskii [5] for free semigroups which extends easily to free SGA:

Proposition 4 *For each system of equations Σ in free SGA with generators C, there is an equation E in free SGA with generators $C \cup c$, $c \notin (C \cup C^{-1})$, such that*

1. *S is a solution of E if and only if S is a solution of Σ.*
2. *$|E| \leq 4|\Sigma|$ and $|E|_V = |\Sigma|_V$.*

Moreover, if the equations in Σ are non-contractible, the E is non-contractible.

Proof. Let $(v_1, w_1), \ldots, (v_n, w_n)$ the system of equations Σ. Define E as

$$(v_1 c v_2 c \cdots c v_n c v_1 c^{-1} v_2 c^{-1} \cdots c^{-1} v_n,\ w_1 c w_2 c \cdots c w_n c w_1 c^{-1} w_2 c^{-1} \cdots c^{-1} w_n).$$

Clearly E is non-contractible because so was each equation (v_i, w_i), and c is a fresh letter. Also if S is a solution of Σ, obviously it is a solution of E. Conversely, if S is a solution of E, then

$$|S(v_1 c v_2 c \cdots c v_n)| = |S(v_1 c^{-1} v_2 c^{-1} \cdots c^{-1} v_n)|,$$

hence

$$|S(v_1 c v_2 c \cdots c v_n)| = |S(w_1 c w_2 c \cdots c w_n)|,$$

and the same for the second pair of expressions with c^{-1}. Now it is easy to show that $S(v_i) = S(w_i)$ for all i: suppose not, for example $|S(v_1)| < |S(w_1)|$. Then $S(w_1)[|S(v_1)| + 1] = c$ and $S(w_1)[|S(v_1)| + 1] = c^{-1}$, impossible. Then argue the same for the rest.

The bounds are simple calculations.

The next result is a very important one, and follows from a straightforward generalization of the result in [7], where it is proved for semigroups.

Theorem 5 *Let E be an equation in free SGA. Then, the exponent of periodicity of a minimal solution of E is bounded by $2^{\mathcal{O}(|E|)}$.*

Proof. It is not worth reproducing here the ten-pages proof in [7] because the changes needed to generalize it to free SGA are minor ones. We will assume that the reader is familiar with the paper [7].

The proof there consist of two independent parts: (1) To obtain from the word equation E a linear Diophantine equation, and (2) To get good bound for it. We will sketch how to do step (1) for free SGA. The rest is completely identical.

First, let us sketch how the system of linear equations is obtained from a word equation E. Let S be a solution of E. Recall that a P-stable presentation of $S(x)$, for a variable x, has the form

$$S(x) = w_0 P^{\mu_1} w_1 P^{\mu_2} \ldots w_{n-1} P^{\mu_{n-1}} w_n.$$

From here, for a suitable P (which is the word that witnesses the exponent of periodicity of $S(E)$), a system of linear Diophantine equations $LD_P(E)$ is built, roughly speaking, by replacing the μ_i by variables x_{μ_i} in the case of variables, plus some other pieces of data. Then it is proved that if S is a minimal solution of E, the solution $x_{\mu_i} = \mu_i$ is a minimal solution of $LD_P(E)$.

For the case of free SGA, the are two key points to note. First, for the variables of the form x^{-1}, the solution $S(x^{-1})$ will have the following P^{-1}-stable presentation (same P, w_i, μ_i as before):

$$S(x^{-1}) = w_n^{-1} (P^{-1})^{\mu_{n-1}} w_{n-1}^{-1} (P^{-1})^{\mu_{n-2}} \ldots w_1^{-1} (P^{-1})^{\mu_1} w_0^{-1}.$$

Second, note that P^{-1} is a subword of PP if and only if P is a subword of $P^{-1}P^{-1}$. Call a repeated occurrence of P in w, say $w = uP^kv$, maximal, if P is neither the suffix of u nor a prefix of v. So it holds that maximal occurrences of P and P^{-1} in w either (1) do not overlap each other, or (2) overlap almost completely (exponents will differ at most by 1).

In case (1), consider the system $LD_P(E') \cup LD_{P^{-1}}(E')$ (each one constructed exactly as in the case of word equations) where E' is the equation E where we consider the pairs of variables x^{-1}, x as independent for the sake of building the system of linear Diophantine equations. And, of course, the variables x_{μ_i} obtained from the same μ_i in $S(x)$ and $S(x^{-1})$ are the same.

In case (2), notice that P-stable and P^{-1}-stable presentations for a variable x differ very little. So it is enough to consider $LD_P(E')$, taking care of using for the P-presentation of $S(x^{-1})$ the same set of Diophantine variables (adding 1 or -1 where it corresponds) used for the P-presentation of $S(x)$.

It must be proved then that if S is a minimal solution of the equation in free SGA E, then the solution $x_{\mu_i} = \mu_i$ is a minimal solution of the corresponding system of linear Diophantine equations defined as above. This can be proved easily with the help of Lemma 2.

Finally, as for the the parameters of the system of Diophantine equations, observe that $|E'| = |E|$, hence the only parameters that grow are the number of variables and equations, and by a factor of at most 2. So the asymptotic bound remains the same as for the case of E', which is $2^{\mathcal{O}(|E|)}$.

The last result concerning equations in free SGA we will prove follows from the trivial observation that every equation in free semigroups is an equation in free SGA. Moreover:

Proposition 6 *Let M be a free semigroup on the set of generators C, and N be a free SGA on the set of generators C, and E an equation in M. Then E is satisfiable in M if and only if it is satisfiable in N.*

Proof. An equation in free SGA which does no contain $(\ )^{-1}$ has a solution if and only if it has a solution which does not contain $(\ )^{-1}$. So the codification of equations in free semigroups into free SGA is straightforward: the same equation.

We get immediately a lower bound for the problem of satisfiability of equations in free SGA by using the corresponding result for the free semigroup case.

Corollary 7 *Satisfiability of equations in free SGA is NP-hard.*

3 Reducing the problem of satisfiability of equations in free groups to satisfiability of equations in free SGA

A *group* is an algebra with a binary associative operation (written as concatenation), a unary operation $(\ )^{-1}$, and a constant 1, with the axioms (1) plus

$$xx^{-1} = 1, \quad x^{-1}x = 1, \quad 1x = x1 = 1. \tag{3}$$

As in the case of free SGA, is not hard to see that the set of non-contractible words over $C \cup C^{-1}$ plus the empty word, and the operations of composition and reverse suitable defined, is a free group with generators C.

Equations in free groups. The formal concept of *equation in free groups* is almost exactly the same as that for free SGA, hence we will not repeat it here. The difference comes when speaking of solutions. A *solution* S of the equation E is (the unique group-homomorphism $S : \mathcal{A} \longrightarrow (C \cup C^{-1})^*$ defined by) a map $S : V \longrightarrow (C \cup C^{-1})^*$ extended by defining $S(c) = c$ for each $c \in C$ and $S(w^{-1}) = (S(w))^{-1}$, which satisfy $S(w_1) = S(w_2)$. Observe that the only difference with the case of SGA is that now we possibly have 'simplifications' of subexpressions of the form ww^{-1} or $w^{-1}w$ to 1, *i.e.* the use of the equations (3).

Proposition 8 (Makanin, cf. Lemma 1.1 in [11]) *For any non-contractible equation E in the free group G with generators C we can construct a finite list $\Sigma_1, \dots, \Sigma_k$ of systems of non-contractible equations in the free SGA G' with generators C such that the following conditions are satisfied:*

1. *E has a non-contractible solution in G if and only if $k > 0$ and some system Σ_j has a non-contractible solution in G'.*
2. *There is $c > 0$ constant such that $|\Sigma_i| \leq |E| + c|E|_V^2$ and $|\Sigma_i|_V \leq c|E|_V^2$ for each $i = 1, \dots, k$.*
3. *There is $c > 0$ constant such that $k \leq (|E|_V)^{c|E|_V^2}$.*

Proof. This is essentially the proof in [11] with the bounds improved. Let E be the equation

$$C_0 X_1 C_1 X_2 \cdots C_{v-1} X_v C_v = 1, \tag{4}$$

where C_i are non-contractible, $v = |E|_V$, and X_i are meta-variables representing the actual variables in E.

Let S be a non-contractible solution of E. By a known result (see [11], p. 486), there is a set W of non-contractible words in the alphabet C, $|W| \leq 2v(2v+1)$, such that each C_i and $S(X_i)$ can be written as a concatenation of no more than $2v$ words in W, and after replacement Equation (4) holds in the free group with generators W.

Let Z be a set of $2v(2v+1)$ fresh variables. Choose words $y_0, x_1, y_1, x_1, \dots, x_v, y_v \in (Z \cup Z^{-1})^*$, each of length at most $2v$, non-contractible, and define the system of equations

1. $C_j = y_j$, $j = 0, \dots, v$,
2. $X_j = x_j$, $j = 1, \dots, v$.

Each such set of equations, for which Equation (4) holds in the free group with generators Z when replacing C_i and X_i by the corresponding words in $(Z \cup Z^{-1})^*$, defines one system Σ_i.

It is clear from the result mentioned earlier, that E has a solution if and only if there is some Σ_i which has a non-contractible solution. How many Σ_i are there? No more than $[(2v(2v+1))^{2v}]^{2v+1}$.

Theorem 9 *For each equation E in a free group G with generators C there is a finite set Q of equations in a free semigroup with anti-involution G' with generators $C \cup \{c_1, c_2\}$, $c_1, c_2 \notin C$, such that the following hold:*

1. *E is satisfiable in G if and only if one of the equations in Q is satisfiable in G'.*
2. *There is $c > 0$ constant, such that for each $E' \in Q$, it holds $|E'| \leq c|E|^2$.*
3. *$|Q| \leq |E|^{c|E|_V^3}$, for $c > 0$ a constant.*

Proof. By Proposition 8, there is a list of systems of non-contractible equations $\Sigma_1, \ldots, \Sigma_k$ which are equivalent to E (w.r.t. non-contractible satisfiability). By Proposition 4, each such system Σ_j is equivalent (w.r.t. to satisfiability) to a non-contractible equation E'. Then, by Proposition 3, for each such non-contractible E', there is a system of equations (now without the restriction of non-contractibility) $\Sigma'_1, \ldots, \Sigma'_{k'}$ such that E' has a non-contractible solution if and only if one of the Σ'_j has a solution (not necessarily non-contractible). Finally, by Proposition 4, for each system Σ', we have an equation E'' which have the same solutions (if any) of Σ'. So we have a finite set of equations (the E'''s) with the property that E is satisfiable in G if and only if one of the E'' is satisfiable in G'.

The bounds in 2. and 3. follow by easy calculations from the bounds in the corresponding results used above.

Remark. It is not difficult to check that the set Q in the previous theorem can be generated non-deterministically in polynomial time.

Corollary 10 *Assume that f_T is an upper bound for the deterministic TIME-complexity of the problem of satisfiability of equations in free SGA. Then*

$$\max\{f_T(c|E|^2),\ |E|^{c|E|_V^3}\},$$

for $c > 0$ a constant, is an upper bound for the deterministic TIME-complexity of the problem of satisfiability of equations in free groups.

4 Conclusions

Our results show that solving equations in free SGA comprises the cases of free groups and free semigroups, the first with an exponential reduction (Theorem 9), and the latter with a linear reduction (Proposition 6). This suggest that free SGA, due to its simplicity, is the 'appropriate' theory to study when seeking algorithms for solving equations in those theories.

In a preliminary version of this paper we stated the following conjectures:

1. Satisfiability of equations in free groups is PSPACE-hard.
2. Satisfiability of equations in free groups is in EXPTIME.
3. Satisfiability of equations in free SGA is decidable.

In the meantime the author proved that satisfiability of equations in free SGA is in PSPACE, hence answering positively (2) and (3). Also independently, Diekert and Hagenah announced the solution of (3) [2].

Acknowledgements. Thanks to Volker Diekert for useful comments.

References

1. V. Diekert, *Makanin's Algorithm for Solving Word Equations with Regular Constraints*, in forthcoming M. Lothaire, *Algebraic Combinatorics on Words.* Report Nr. 1998/02, Fakultät Informatik, Universität Stuttgart.
2. V. Diekert, Personal communication, 8 Oct. 1999.
3. C. Gutiérrez, *Satisfiability of Word Equations with Constants is in Exponential Space*, in Proc. FOCS 98.
4. C. Gutiérrez, *Solving Equations in Strings: On Makanin's Algorithm*, in Proceedings of the Third Latin American Symposium on Theoretical Informatics, LATIN'98, Campinas, Brazil, 1998. In LNCS 1380, pp. 358-373.
5. J.I. Hmelevskii, *Equations in a free semigroup*, Trudy Mat. Inst. Steklov 107(1971), English translation: Proc. Steklov Inst. Math. 107(1971).
6. J. Jaffar, *Minimal and Complete Word Unification*, Journal of the ACM, Vol. 37, No.1, January 1990, pp. 47-85.
7. A. Kościelski, L. Pacholski, *Complexity of Makanin's algorithm*, J. Assoc. Comput. Mach. 43 (1996) 670-684.
8. A. Kościelski, L. Pacholski, *Makanin's algorithm is not primitive recursive*, Theoretical Computer Science 191 (1998) 145-156.
9. Lothaire, M. *Combinatorics on Words*, Cambridge Mathematical Texts, reprinted 1998.
10. G.S. Makanin, *The problem of satisfiability of equations in a free semigroup*, Mat. Sbornik 103, 147-236 (in Russian). English translation in Math. USSR Sbornik 32, 129-198.
11. G.S. Makanin. *Equations in a free group*. Izvestiya NA SSSR 46, 1199-1273, 1982 (in Russian). English translation in Math USSR Izvestiya, Vol. 21 (1983), No. 3.
12. W. Rytter and W. Plandowski, *Applications of Lempel-Ziv encodings to the solution of word equations*, In Proceedings of the 25th. ICALP, 1998.
13. Plandowski, W., *Satisfiability of word equations with constants is in NEXPTIME*, in Proc. STOC'99.
14. Plandowski, W., *Satisfiability of word equations with constants is in PSPACE*, in Proc. FOCS'99.
15. A.A. Razborov, *On systems of equations in a free group*, Izvestiya AN SSSR 48 (1984) 779-832 (in Russian). English translation in Math. USSR Izvestiya 25 (1985) 115-162.
16. K. Schulz, *Word Unification and Transformation of Generalized Equations*, Journal of Automated Reasoning 11:149-184, 1993.