

Complexity of the bisection method

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Abstract

The *bisection method* is the consecutive bisection of a triangle by the median of the longest side. In this paper we prove a subexponential asymptotic upper bound for the number of similarity classes of triangles generated on a mesh obtained by iterative bisection, which previously was known only to be finite. The relevant parameter is γ/σ , where γ is the biggest and σ is the smallest angle of the triangle. We get this result by introducing a taxonomy of triangles that precisely captures the behaviour of the bisection method. We also prove that the number of directions on the plane given by the sides of the triangles generated is finite. Additionally, we give purely geometrical and intuitive proofs of classical results for the bisection method.

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1. Introduction

Adaptive finite element methods and multigrid algorithms are widely used numerical techniques for the accurate analysis of physical and engineering problems modelled by partial differential equations, both of which require sequences of flexible and quality discretizations of the associated geometric region. Longest edge refinement algorithms for triangulations, based on the (longest edge) bisection of triangles [6,5] were especially designed to deal with these issues: they are able to perform iterative local refinement of the involved triangulations by essentially maintaining the geometrical quality of the input mesh as needed in finite element applications; they produce hierarchies of quality and nested irregular triangulations as required for non-structured multigrid methods. Furthermore because of their simplicity, these algorithms have been also successfully used for the parallel refinement of big meshes in parallel finite element applications [3,6].

Longest edge algorithms work as follows: they perform selective and iterative longest edge bisection of some target triangles and some neighbours in order to produce a conforming mesh (where the intersection of pairs of neighbour triangles is either a common vertex or a common edge). For an illustration of the use of these algorithms, see Fig. 1, where a locally refined triangulation of an input triangulation is shown. The properties of longest-edge algorithms are inherited from the non-degeneracy properties of the iterative longest edge bisection of triangles, which essentially

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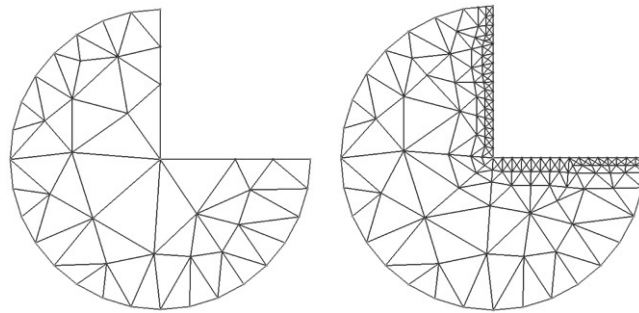


Fig. 1. An initial triangulation (on the left), and a local refinement of it.

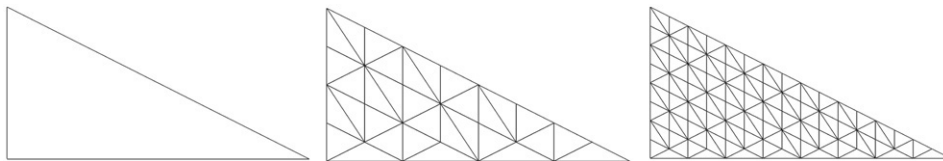


Fig. 2. Refinement by iterated bisections.

guarantee that consecutive bisections of the triangles nested in any triangle t of smallest angle σ produce triangles t' (of minimum angle σ') such that $\sigma' \geq \sigma/2$, and that the number of non-similar triangles generated is finite. See the example in Fig. 2. These properties not only assure that the smallest angles of the refined triangles are bounded, but also guarantee that the refined area is increasingly covered by better triangles as the refinement proceeds. Note, however, that this paper does not study refinement algorithms but the basic properties of the longest edge bisection of an individual triangle (bisection method) throughout the bisection levels.

The systematic study of the bisection method began in a series of papers [2,7–9,1] around two decades ago. First, Rosenberg and Stenger [7] proved that the method does not degenerate the smallest angle of the triangles generated by showing that it does not decrease beyond $\sigma/2$, where σ is the smallest angle from the initial triangle. Then Kearfott [2] proved a bound on the behaviour of the *diameter* (the length of the longest side of any triangle obtained). Later Stynes [8] presented a better bound for certain triangles. This bound was improved independently by Stynes [9] and Adler [1] for all triangles. From their proofs they also deduced that the number of classes of similarity of triangles generated is finite, although they give no bound.

There is very little research so far on complexity aspects of the bisection method. Although it is known that different types of triangles behave radically differently under iterative bisection (“good” and “bad” triangles), no systematic classification of them is known.

This paper attempts to fill these gaps in the analysis of the bisection method. We present a precise taxonomy of triangles that captures the behaviour of the bisection method for different types of triangles. We introduce as our main parameter the smallest angle, and prove that in the plane it predicts faithfully the behaviour of the bisection method. We use this framework to prove new results and to give intuitive proofs of classical results.

The contributions of this paper are as follows:

- A taxonomy of triangles reflecting the behaviour of the bisection method. We consider six classes of triangles, and two main groups.
- An asymptotic bound on the number of non-similar triangles generated. We prove a subexponential asymptotic upper bound, identify the instances where this bound is polynomial, and describe worst-case instances.
- An analysis of lower bounds on the smallest angle of triangles in the mesh obtained using the bisection method for each class of triangles defined.
- A proof that there is a finite number of directions in the plane generated by the corresponding segments (sides) of the triangles generated, and asymptotic bounds on this number.

Additionally, we present a unified view of the main known results for the bisection method from an elementary geometry point of view. This approach allows intuitive proofs and has the advantage of presenting the geometry inherent in the method.

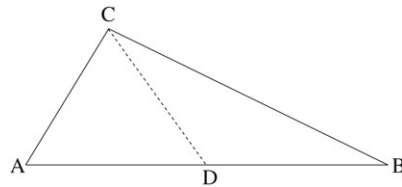


Fig. 3. Triangle ABC with $AB \geq BC \geq CA$. D is the midpoint of AB .

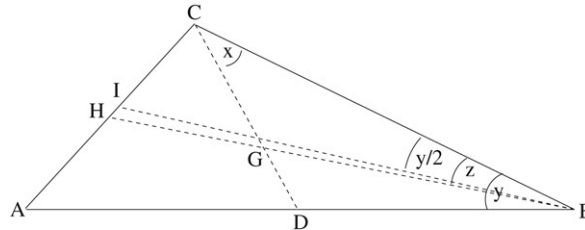


Fig. 4. BI is bisector, BH and CD are medians, G is center of gravity.

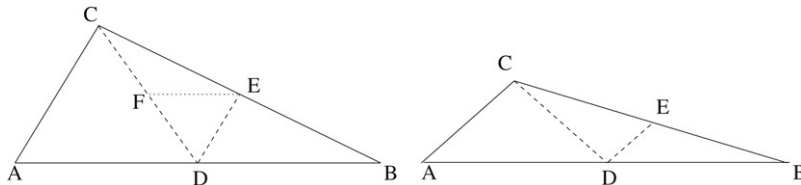


Fig. 5. Steps: Of type A on the left when $CD \geq CE$, and of type B on the right when $CD \leq CE$. Vertices D , E and F are midpoints of the corresponding segments.

2. Notation and preliminaries

Capital letters denote points on the plane. In order to simplify we will avoid extra symbols and sometimes overload some notations. AB denotes a segment as well as the length of this segment usually denoted by \overline{AB} . An angle $\angle ACB$ denotes the actual instance as well as the value (measure) of it. A circle of centre A and radius r is denoted by $C(A, r)$.

A *bisection*, by the median of the longest side, of triangle ABC with $AB \geq BC \geq CA$, is the figure obtained by tracing the segment CD , where D is the midpoint of the longest segment AB . See Fig. 3. We will study the properties obtained by successively bisecting the triangles so obtained.

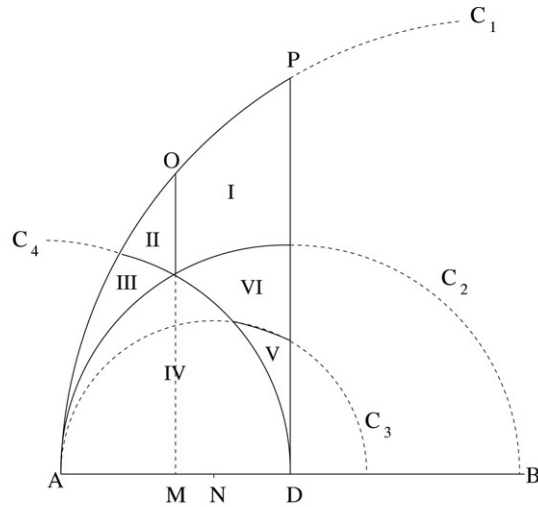
For a given triangle PQR , denote by σ_{PQR} (respectively γ_{PQR}) the value of the smallest (respectively greatest) angle in triangle PQR , and by β_{PQR} the remaining angle.

We will need a simple and useful technical lemma:

Lemma 1. For $\triangle ABC$ with $AB \geq BC \geq CA$, it holds that $\angle BCD \geq \frac{1}{2} \angle DBC$.

Proof (See Fig. 4). Let ABC be a triangle with $AB \geq BC \geq CA$, let BI be the bisector of $\angle ABC$, let BH and CD be medians, and let G be its center of gravity. From $AB \geq CA$ and elementary geometry it follows that $BG \geq GC$, hence $x \geq z \geq y/2$. Note that $x = y/2$ if and only if $AB = CA$. \odot

To simplify the study of the bisection method, it is convenient to group two or three consecutive partial bisections in triangle ABC , in what we will call a *step*, as follows. For this discussion refer to Fig. 5. Let E be the midpoint of segment BC . Note that if $CD \geq CE$, then CD , DE and EF are consecutive bisections by the median of the longest side, and after these bisections we get exactly three new non-similar triangles: ADC , CDE and CDB . Furthermore, note that every succeeding triangle in the interior of triangle CDB is similar to one of these preceding triangles (see left side of Fig. 5). When this happens, we call these three consecutive bisections a *step of type A*. Note that $\triangle ADC$ is the only triangle that possibly could generate new triangles not similar to those previously generated.



Region	Defining properties	Other properties	step type
I	$AD \leq CD \leq CA$	$\gamma \leq \pi/2$	A
II	$AD \leq CA \leq CD$	$\gamma \leq \pi/2$	A
III	$CA \leq AD \leq CD$	$\gamma \leq \pi/2$	A
IV	$CA, CD \leq AD$	$\gamma \geq \pi/2$	A/B
V	$AD \leq CA; CD \leq CE$	$\gamma > \pi/2$	B
VI	$CD \leq AD \leq CA; CD \geq CE$	$\gamma \geq \pi/2$	A

Fig. 6. Regions defining classes of triangles ABC . A (virtual) vertex C lying in one of the regions defines a triangle ABC with longest side AB , and greatest angle $\angle ACB$ (denoted γ).

On the other hand, if $CD \leq CE$, the longest side in triangle CDE is now CE . In this case after two bisections (CD and DE), we get two new triangles not similar to previous ones, namely ADC and CDE (see right side of Fig. 5). We call these two consecutive bisections a *step of type B*. Note that for type B bisections, triangles ADC and CDE are the only triangles that could generate new triangles not similar to the previous ones.

3. A classification of triangles

The behaviour of the bisection method depends on the type of triangle to be bisected. We will partition the set of all triangles in classes that reflect this behaviour by considering some elementary geometrical properties. The starting point will be a triangle ABC as in Fig. 3.

The analysis is based on the geometrical positions where vertex C of triangle ABC may lie, assuming $AB \geq BC \geq CA$. For this discussion, we refer to Fig. 6, where AB represents the longest side of the hypothetical triangle, D the midpoint of AB , M is the midpoint of AD , N is such that $AN = AB/3$, $MO \perp AB$ and $DP \perp AB$. Arcs C_1, C_2, C_3 and C_4 belong, respectively, to circles $C(B, \overline{AB}), C(D, \overline{AD}), C(N, \overline{AN}),$ and $C(A, \overline{AD})$.

From the condition $AB \geq BC \geq CA$, it follows that vertex C of a triangle with base AB must be in the region bounded by arc AP and straight lines PD and AD . We partition this region into six subregions, denoted by Roman numerals, with the property that triangles in the same subregion present similar behavior with regard to bisection by the median of the longest side, as stated in Lemma 2. Note that arc C_3 is the set of points C for which $BC = 2CD$, and is precisely the geometrical point which separates those triangles for which steps of type A apply from those triangles for which steps of type B apply. The table in Fig. 6 lists the defining properties of triangles in each region.

Let us consider the process of bisecting iteratively a triangle. We will proceed following steps of type A or B, as follows:

- (1) Perform a step of the corresponding type (depending on the triangle);

- (2) From the triangles obtained, choose nondeterministically a triangle not similar to any previously generated. If there is no such triangle (i.e. all triangles generated are similar to previous ones), stop; else goto 1.

Lemma 2. *Let ABC be a triangle. For the iterative process described above it holds:*

- (1) *If C is in region I, it generates at most 4 non-similar triangles as shown in Fig. 8.*
- (2) *If C is in region II, it generates at most 5 non-similar triangles.*
- (3) *If C is in region III, new $\triangle ADC$ belongs either to regions II or III. Moreover, after no more than $\lceil 5.7 \log(\frac{\pi}{6\sigma}) \rceil$ steps the only new triangle generated not similar to any previously generated belongs to region II.*
- (4) *If C is in region IV or V, after no more than $\lceil (\gamma - \pi/2)/\sigma \rceil$ steps, the only new triangle not similar to any previously generated has $\gamma \leq \pi/2$ (i.e. belongs to region I, II or III.)*
- (5) *If C is in region VI, new $\triangle ADC$ belongs to region I.*

Proof. 1. The proof follows from the analysis of the relationships between the sides of the triangles generated. See the definition of region I and Fig. 8. Observe that the new triangles generated are $\triangle ADC$ and $\triangle BCD$ (after the initial bisection). On bisecting $\triangle ADC$, $\triangle DCH$ is another new triangle, but $\triangle ADH$ is similar to $\triangle ABC$. On bisecting $\triangle DCH$, both $\triangle HDF$ and $\triangle CHF$ are similar to earlier triangles.

2. Consider the reflection C' of point C across the line MO . Then triangle ABC' belongs to region I, and $\triangle ADC'$ is congruent to $\triangle ADC$. Because triangle ABC is in region II, a step of type A applies, hence – modulo similarity – the triangles generated by $\triangle ABC$ are triangle BCD plus those generated by triangle ADC , which are included in those generated by triangle ABC' (at most four).

3. By the arguments in Section 2, at each step we may generate at most 3 new (non-similar) triangles, and one triangle ADC which must be recursively bisected. First, observe that $\triangle ADC$ has $\gamma \leq \pi/2$, and $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$ (due to the equality $\sigma_{ADC} = \angle ADC$ and to Lemma 1). By basic geometry we can show that $\triangle ADC$ lies in region I, II, or III. Now, because at each step σ is increased by $3/2$, it is enough to find the smallest k such that $(\frac{3}{2})^k \sigma \geq \pi/6$, that is, $k \geq \log(\frac{\pi}{6\sigma}) / \log(3/2)$. After this number of steps, the triangle being bisected recursively must lie in region II. The solution, denoted by $k(\sigma)$, is $k(\sigma) = \lceil 5.7 \log(\frac{\pi}{6\sigma}) \rceil$.

4. There are two cases depending on whether step A or step B is applied (whether $CD \leq CE$ or not). If step A applies then after that step, there is one triangle ADC to be bisected recursively. By Lemma 1 the greatest angle of $\triangle ADC$ is at most $\gamma - \sigma_{ABC}$. If step B applies then after one step, the only new triangles generated not similar to any previous triangle, $\triangle ADC$ and $\triangle CDE$, each have a greatest angle which is less than γ_{ABC} by σ_{ABC} . Hence it is enough to find the smallest k such that $\gamma - k\sigma \leq \pi/2$. The solution depends on two parameters and is $\lceil (\gamma - \pi/2)/\sigma \rceil$.

5. Just observe that $\gamma_{ADC} \leq \pi/2$ and σ_{ADC} is the same as $\angle CAB$ of $\triangle ABC$. \odot

4. Number of similarity classes of triangles

We are ready to prove the main theorem:

Theorem 1. *Let ABC a triangle, σ its smallest angle, and γ its biggest angle:*

- (1) *The number of steps to be executed by the bisection method until no more non-similar triangles are generated is $\mathcal{O}(\sigma^{-1})$;*
- (2) *If C is above arc C_3 , then the number of non-similar triangles generated by the bisection method is $\mathcal{O}(\log(\sigma^{-1}))$;*
- (3) *The number of non-similar triangles generated by the bisection method is asymptotically bounded by a subexponential function in γ/σ , i.e., it is $\mathcal{O}(a^{(\gamma/\sigma)^c})$ for any $a > 1$ and $c > 0$.*

Proof. The proof is based on two facts: (1) In the bisection method, the new triangles generated which are not similar to any previously generated triangles (and which may lead to the generation of more non-similar triangles) follow the paths in the graph shown in Fig. 7 as proved in Lemma 2 (possibly staying in a node several steps). (2) In the counting process of the number of steps and non-similar triangles generated, using the remark at the end of Section 2, it follows that for a given triangle ABC (refer to Fig. 5), one only needs to examine the behaviour of triangle ADC and, for regions IV and V, also the triangle CDE . With these observations in mind, let us prove the statements of the Theorem.

1. Let us calculate the maximum number of steps to be executed before arriving at region I in the worst case. This occurs for triangles in regions IV or V. From Lemma 2 it follows that a rough upper bound on the number of steps

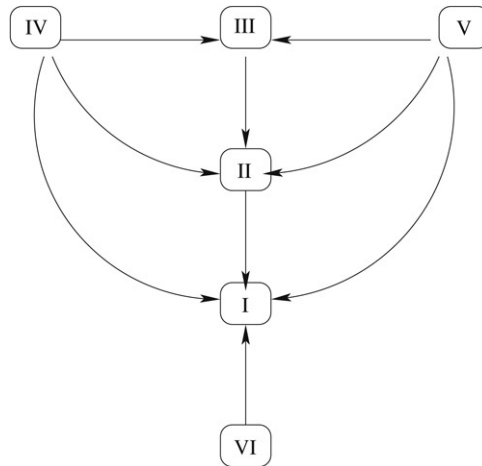


Fig. 7. A directed graph illustrating Lemma 2. Arcs indicate possible jumps in our proof, where the triangle being bisected recursively jumps from one region to another.

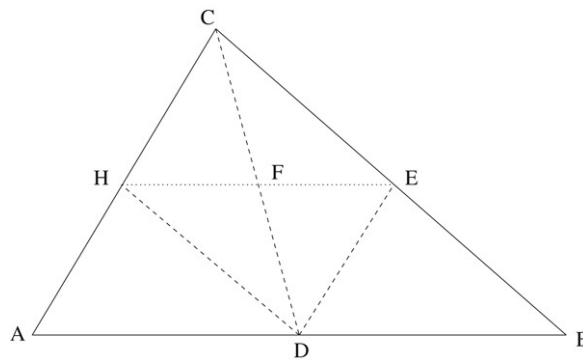


Fig. 8. After bisections in a triangle in Region I.

executed is given by the sum $2 + \lceil 5.7 \log(\frac{\pi}{6\sigma}) \rceil + \lceil (\gamma - \pi/2)/\sigma \rceil$. This number is asymptotically linear in σ^{-1} because $\pi/3 \leq \gamma < \pi$. (Compare [9], Cor. 1.)

2. For a triangle ABC above arc C_3 , the number $N(ABC)$ of non-similar triangles is $N(ABC) \leq 3 + N(ADC)$, where the number 3 corresponds to $\triangle ABC$, $\triangle DBC$ and $\triangle DEC$. The statement follows from Lemma 2, items 1, 2 and 3, and the remarks of the end of Section 2.

3. The complex case is region IV. (The analysis for region V is similar.) Here $N(ABC) \leq 2 + N(ADC) + N(CDE)$, as follows from the discussion at the end of Section 2. First let us prove that $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$. If C is to the left of MO , then σ_{ADC} is the angle $\angle ADC$ and by Lemma 1 we are done. Next consider the geometric position of the set of points C such that $\beta_{ABC} = \frac{3}{2}\sigma_{ABC}$. This is a line L passing through D with negative slope. If C lies to the right of L , then $\triangle ADC$ will be in region IV to the left of MO and we are in the previous case in one step. If C lies in between L and MO , then $\sigma_{ADC} = \angle CAD = \beta_{ABC} \geq \frac{3}{2}\sigma_{ABC}$ by definition.

Now, using the fact that both triangles ADC and CDE have γ diminished by σ , the fact already proved that $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$, and observing that $\sigma_{DBC} \geq \sigma_{ABC}$, we have the following recurrence equation for the number $N(\gamma, \sigma)$ of non-similar triangles generated:

$$N(\gamma, \sigma) \leq N\left(\gamma - \sigma, \frac{3}{2}\sigma\right) + N(\gamma - \sigma, \sigma).$$

Observe that the equation holds for triangles in region IV, i.e. for $\gamma > \pi/2$. Putting $N(\gamma, \sigma) = f(\gamma/\sigma)$ and $n = \lceil \gamma/\sigma \rceil$, the recurrence essentially reduces to $f(n) = f(2n/3) + f(n - 1)$. This recurrence has no polynomial solution, but has a subexponential asymptotic upper bound, that is, for each $a > 1$ and $c > 0$, $f(n) \leq a^{n^c}$ for

sufficiently large n . To prove this, just assume inductively $f(\frac{2}{3}n) \leq a^{(\frac{2}{3}n)^c}$ and $f(n - 1) \leq a^{(n-1)^c}$ and check that there is $\epsilon > 0, n_\epsilon > 0$ such that for $n \geq n_\epsilon, f(n) = f(\frac{2}{3}n) + f(n - 1) \leq (a^{((\frac{2}{3}n)^c - 1)n^c} + a^{-\epsilon})a^{n^c} \leq a^{cn}$. \odot

It is interesting to note that not only the number of non-similar triangles generated by the bisection method is finite, but a stronger result can be proved:

Proposition 1. *The bisection method generates a finite number of different directions in the plane. Moreover, in the worst case this number is asymptotically bounded by a subexponential function in γ/σ , i.e. it is $\mathcal{O}(a^{(\gamma/\sigma)^c})$ for any $a > 1$ and $c > 0$.*

Proof. The crucial and simple observation is that the directions of the sides of the triangles generated which are similar to previously generated ones, coincide with already generated directions. Using this fact, the bound follows from Theorem 1 and the analysis of the regions: at each step only one new direction is added except in regions IV and V where the number of directions is (possibly) doubled. \odot

5. Classical results revisited

Using only elementary geometric methods it is possible to re-prove classical results about the smallest angle and the diameter of a mesh obtained by iterative bisection. Let us recall the classical terminology. Triangles obtained by successive bisection of a triangle are grouped into *levels*. Level 0 is the original triangle. Inductively, level $n + 1$ are all triangles obtained by bisecting all triangles of level n . The *diameter* of a set of triangles is the length of the longest side of any triangle in the set.

Theorem 2. (1) *The bisection method gives $\mu_{ABC} \geq \frac{1}{2}\sigma_{ABC}$, where μ_{ABC} is the smallest angle of any triangle in any level. For triangles below arc C_2 it holds that $\mu_{ABC} = \sigma_{ABC}$.*
 (2) *Define d_j as the diameter of triangles in level j . Then, (i) $d_5 \leq d_0/2$, i.e. after 5 bisection levels, the diameter of the generated mesh is no more than half of the original, and (ii) $d_j \leq c2^{-j/2}d_0$, where c is a small constant depending on the regions.*

Proof. 1. It is enough to analyse the behaviour of bisection steps (as defined at the end of Section 2). We will prove that the only case when the smallest angle diminishes in a bisection step occurs for triangles in region I (see Fig. 8), being the worst case when $C = P$, that is, equilateral triangles. So let us show that for a triangle ABC belonging to any region different from I, the smallest angle of the triangles generated from triangle ABC is bigger than σ_{ABC} . First, by a case by case check, it follows that for triangles in regions below C_2 always holds $\sigma_{ADC} > \sigma_{ABC}$ and $\sigma_{DBC} > \sigma_{ABC}$. Second, for triangles in region III, the new triangle ADC has $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$ (because $\sigma_{ADC} = \angle ADC$ and Lemma 1), and clearly $\sigma_{DBC} > \sigma_{ABC}$. For triangles ABC in region II, observe that $\sigma_{ABC} \leq \pi/6$ and $\sigma_{ADC} = \angle ACD > \sigma_{ABC}$.

2. The item (i) can be easily checked, the worst case being triangles in region I.

As for (ii), using formula the area of a triangle $A = \frac{1}{2}bh$ and the fact that the area decreases exactly by half after a bisection, one gets immediately $b_j = (\frac{h_0}{h_j})\frac{b_0}{2^j}$, where the sub-indexes indicate bases and heights corresponding to triangles in the j -th level.

Now the crucial point is to observe that: (i) for triangles whose vertex C is below arcs C_2 or C_4 , the diameter decreases by half after two levels, i.e. $d_2 \leq d_0/2$; and (ii) the fact we already know that, as bisection progresses, triangles “go up” the boundary of arcs C_4 and C_2 . Hence, h_j can be bound (in terms of b_j), because from the fact mentioned above, we can deduce that σ_j is no smaller than say $\pi/7$. Similarly, h_0 has a fixed bound in terms of b_0 (the worst case being $\sqrt{3}b_0/2$). Using these formulae we get $b_j^2 \leq c^2b_0^22^{-j}$, for some constant $c \leq \sqrt{3}$ (cf. also [1]). From here, taking the square root we get the statement of the theorem. \odot

6. Conclusion

We presented a taxonomy of triangles in the plane which captures the behavior of the bisection method. Besides allowing us to prove complexity results for the bisection method, this classification is useful to refine bounds for each class of triangles, and to determine more precisely lower bounds on the smallest angle μ_{ABC} in the mesh, as well as

the number of non-similar triangles generated. The analysis could be further refined considering regions we did not separate, e.g. below arc C_2 , above arc C_3 and to the left of MO in Fig. 6.

Further work includes use of this theoretical analysis to refine algorithms of bisection (4-edge partition, simple bisection, etc.) according to the type of triangle found in each iteration. We hope the methodology can shed light on the extension of the bisection method to the 3D case, whose main questions are still unanswered [4].

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