A Geometric Approach to the Bisection Method

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Abstract. The *bisection method* is the consecutive bisection of a triangle by the median of the longest side. This paper introduces a taxonomy of triangles that precisely captures the behavior of the bisection method. Our main result is an asymptotic upper bound for the number of similarity classes of triangles generated on a mesh obtained by iterative bisection, which previously was known only to be finite. We also prove that the number of directions on the plane given by the sides of the triangles generated is finite. Additionally, we give purely geometric and intuitive proofs of classical results for the bisection method.

1 Introduction

Longest-side bisection algorithms for the refinement of 2-dimensional triangulations were developed to fill a gap in the design of adaptive software for finite element applications to analyze physical problems described by partial differential equations, where the availability of algorithms able to produce automatic and local refinement of the mesh is crucial. A discussion of the algorithms and some generalizations can be found in [4, 5]. These algorithms were designed to take advantage of the non-degeneracy properties of the iterative longest-side bisection (bisection method) of triangles, which essentially guarantee that consecutive bisections of the triangles nested in any triangle t_0 of smallest angle σ_0 produce triangles t (of minimum angle σ_t) such that $\sigma_t \geq \sigma_0/2$, and where the number of non-similar triangles generated is finite.

The systematic study of the bisection method began in a series of papers [2, 7–9, 1] around two decades ago. First, Rosenberg and Stenger [7] proved that the method does not degenerate the smallest angle of the triangles generated by showing that it does not decrease beyond $\sigma/2$, where σ is the smallest angle from the triangle we started.

Then Kearfott [2] proved a bound on the behavior of the *diameter* (the length of the longest side of any triangle obtained). In [8] a better bound was presented for certain triangles. This bound was improved independently by Stynes [9] and Adler [1] for all triangles. From their proofs they also deduced that the number of classes of similarity of triangles generated is finite, although they give no bound.

There is very little research so far on complexity aspects of the bisection method. Although it is known that different types of triangles behave radically different under iterative bisection ("good" and "bad" triangles), no systematic classification of them is known.

This paper attempts to fill these gaps in the analysis of the bisection method. We present a precise taxonomy that captures the behavior of the bisection method for different types of triangles. We introduce as main parameter the smallest angle and prove that in the plane it predicts faithfully the behavior of the bisection method. We use this framework to prove new results and to give intuitive proofs of classical results.

The contributions of this paper are as follows:

- A taxonomy of triangles reflecting the behavior of the bisection method. We consider six classes of triangles, and two main groups.
- An asymptotic bound on the number of non-similar triangles generated. We prove a super-polynomial upper bound, identify the instances where this bound is polynomial, and describe worst case instances.
- An analysis of lower bounds on the smallest angle of triangles in the mesh obtained using the bisection method for each class of triangles defined.
- A proof that there is a finite number of directions in the plane generated by the corresponding segments (sides) of the triangles generated, and asymptotic bounds on this number.

Additionally, we present a unified view of the main known results for the bisection method from an elementary geometry point of view. This approach allows intuitive proofs and has the advantage of presenting the geometry inherent to the method.

2 Notation and preliminaries

Capital letters denote points on the plane. In order to simplify we will avoid extra symbols and sometimes overload some notations. AB denotes a segment as well as the length of this segment usually denoted by $\overline{AB}$. An angle $\angle ACB$ denotes the actual instance as well as the value (measure) of it. A circumference of center A and radius r is denoted by $C(A, r)$.

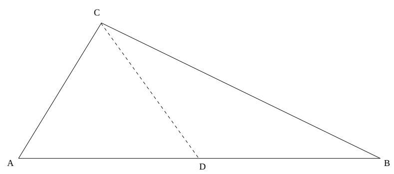

Fig. 1. Triangle ABC with $AB \geq BC \geq CA$. D is the midpoint of AB.

A *bisection*, by the median of the longest side, of triangle ABC with $AB \geq BC \geq CA$, is the figure obtained by tracing the segment CD, where D is the midpoint of the longest segment AB. See Figure 1. We will study the properties obtained by successively bisecting the triangles so obtained.

For a given triangle PQR, denote by σ_{PQR} (respectively γ_{PQR}) the value of the smallest (respectively greatest) angle in triangle PQR, and by β_{PQR} the remaining angle.

We will need a simple and useful technical lemma:

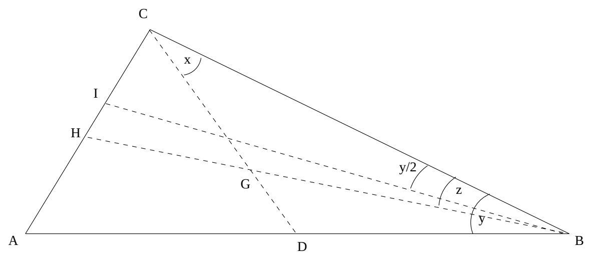

Fig. 2. BI is bisectriz, BH and CD are medians, G is center of gravity.

Lemma 1. *For $\triangle ABC$ with $AB \geq BC \geq CA$, it holds $\angle BCD \geq \frac{1}{2}\angle DBC$.*

Proof. (See Figure 2.) Let be ABC a triangle with $AB \geq BC \geq CA$, let BI the bisectriz of $\angle ABC$), let BH and CD be medians, and let G be its center of gravity. From $AB \geq BC \geq CA$ and elementary geometry it follows that $BG \geq GC$, hence $x \geq z \geq y/2$. Note that $x = y/2$ if only if $AB = AC$.

To simplify the study of the bisection method, it is convenient to group two or three consecutive bisections in triangle ABC, in what we will call a *step*, as follows. For this discussion refer to Figure 3. Let E be the middle-point of segment CB. Note that if $CD \geq CE$, then CD, DE and EF are consecutive bisections by the median of the longest side, and after these bisections we get exactly three non-similar triangles: ADC, CDE and CDB (all others are similar to one of these, see left side of Figure 3). We call these three consecutive bisections a *step of type A*. Note that $\triangle ADC$ is the only triangle that possibly generates new triangles non-similar to already generated ones.

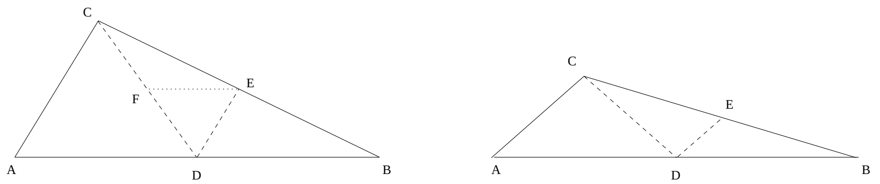

Fig. 3. Steps: Of type A on the left when $CD \geq CE$, and of type B on the right when $CD \leq CE$. Vertices D, E and F are midpoints of the corresponding segments.

On the other hand, if $CD \leq CE$, the longest side in triangle CDE is now CE. Hence we bisect only twice (CD and DE) and get two new triangles, namely ADC and CDE (see right side of Figure 3). We call these two consecutive bisections a *step of type B*. Note that for type B bisections, triangles ADC and CDE are the only triangles that could generate new triangles non-similar to already generated ones.

3 A classification of triangles

The behavior of the bisection method depends on the type of triangle to be bisected. We will partition the set of all triangles in classes that reflect this behavior by considering some elementary geometrical properties. The starting point will be a triangle ABC as in Figure 1.

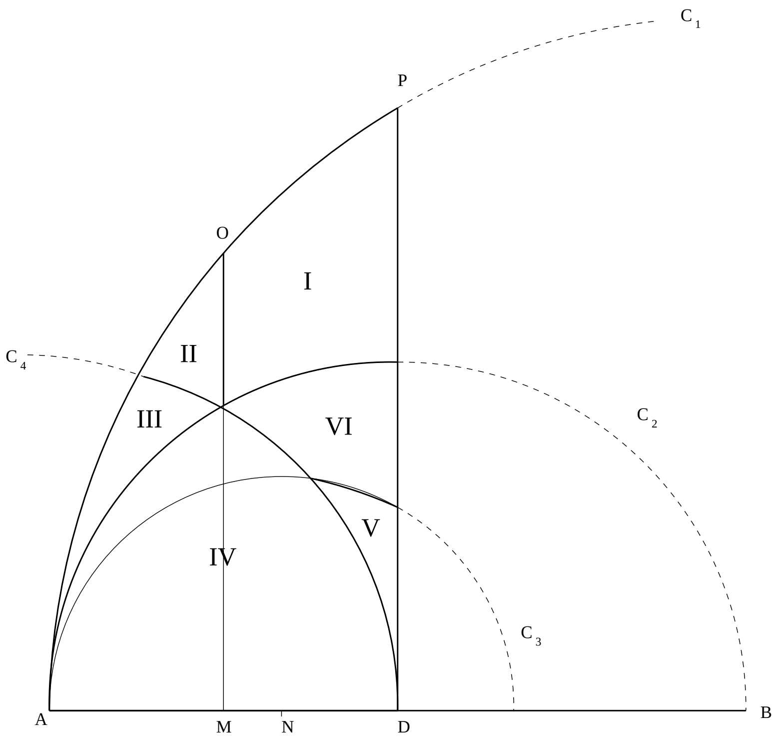

Fig. 4. Regions

Region	Defining properties	Other properties	step type
I	$AD \leq CD \leq AC$	$\gamma \leq \pi/2$	A
II	$AD \leq AC \leq CD$	$\gamma \leq \pi/2$	A
III	$AC \leq AD \leq CD$	$\gamma \leq \pi/2$	A
IV	$AC, CD \leq AD$	$\gamma \geq \pi/2$	A/B
V	$AD \leq AC$; $CD \leq CE$	$\gamma > \pi/2$	B
VI	$CD \leq AD \leq AC$; $CD \geq CE$	$\gamma \geq \pi/2$	A

The analysis is based on the geometrical places where vertex C of triangle ABC lies, assuming $AB \geq CB \geq CA$. For this discussion, we refer to Figure 4, where AB represents the longest side of the hypothetical triangle, D the midpoint of AB, M is the midpoint of AD, N is such that $AN = AB/3$, $MO \perp AB$ and $DP \perp AB$. The arc C_1 belongs to a circumference $C(B, \overline{AB})$, arc C_2 to $C(D, \overline{AD})$, arc C_3 to $C(N, \overline{AN})$ and finally arc C_4 to $C(A, \overline{AD})$.

From the condition $AB \geq BC \geq CA$, it follows that vertex C of a triangle with base AB must be in the region bounded by arc AP and lines PD and AD. We partition this region into six subregions, denoted by Roman numerals, with the property that triangles in the same subregion present similar behavior with regard to bisection by the median of the longest side, as stated in Lemma 2. Note that arc C_3 is the set of points C for which $CD = CE$, and is precisely the geometrical place which separates those triangles for which steps of type A apply from those triangles for which steps of type B apply. Table in page 5 lists defining properties of triangles in each region.

Let us consider the process of bisecting iteratively a triangle. In what follows by a "new triangle" we mean a triangle not similar to one already generated. We will proceed following steps of type A or B, as follows:

1. Perform a step of the corresponding type (depending on the triangle);
2. Choose nondeterministically one of the new triangles obtained. If there is no such triangle (i.e. all triangles generated are similar to previous ones), stop; else goto 1.

Lemma 2. *Let ABC be a triangle. For the iterative process described above it holds:*

1. *If C is in region I, it generates at most 4 non-similar triangles as shown in Figure 5, all of them belonging to region I.*
2. *If C is in region II, new $\triangle ADC$ belongs to region I.*
3. *If C is in region III, new $\triangle ADC$ belongs either to regions II or III. Moreover, in no more than $\lceil 5.7 \log(\frac{\pi}{6\sigma}) \rceil$ steps the only new triangle generated belongs to region II.*
4. *If C is in region IV or V, after no more than $\lceil (\gamma - \pi/2)/\sigma \rceil$ steps, the only new triangle has $\gamma \leq \pi/2$ (i.e. belongs to region I, II or III.)*
5. *If C is in region VI, new $\triangle ADC$ belongs to region I.*

Proof. 1. Follows from the analysis of the relations among sides of the triangles generated. See definition of region I and Figure 5.

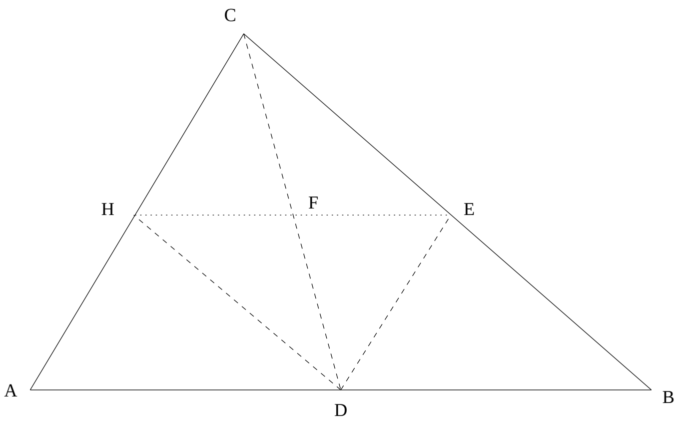

Fig. 5. After bisections in a triangle in Region I

2. Consider the triangle ABC' in region I, where C' is the reflex of C on the line MO. We know that $\triangle ADC'$ is in region I. Now observe that triangle $\triangle ADC$ is congruent to $\triangle ADC'$.

3. First, observe that $\triangle ADC$ has $\gamma \leq \pi/2$, and $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$ (because $\sigma_{ADC} = \angle ADC$ and Lemma 1). Now, because at each step σ is increased by 3/2, it is enough to find the smallest k such that $(\frac{3}{2})^k\sigma \geq \pi/6$, that is, $k \geq \log(\frac{\pi}{6\sigma})/\log(3/2)$. The solution, denoted by $k(\sigma)$, is $k(\sigma) = \lceil 5.7\log(\frac{\pi}{6\sigma})\rceil$.

4. After one step, the only new triangles generated, $\triangle ADC$ and $\triangle CDE$, decrease their greatest angle by σ_{ABC}. Hence it is enough to find the smallest k such that $\gamma - k\sigma \leq \pi/2$. The solution depends on two parameters and is $\lceil(\gamma - \pi/2)/\sigma\rceil$.

5. Just observe that $\gamma_{ADC} \leq \pi/2$ and σ_{ADC} is the same as $\angle CAB$ of $\triangle ABC$.

4 Number of similarity classes of triangles

We are ready to prove the main theorem:

Theorem 1. *Let ABC a triangle and σ its smallest angle.*

1. *The number of steps to be executed by the bisection method until no more non-similar triangles are generated is $\mathcal{O}(\sigma^{-1})$*
2. *If C is above arc C_3, then the number of non similar triangles generated by the bisection method is $\mathcal{O}(\log(\sigma^{-1}))$*
3. *The number of non similar triangles generated by the bisection method is $\mathcal{O}(\sigma^{\log\sigma})$.*

Proof. 1. Let us calculate the maximum number of steps to be executed before arriving to region I in the worst case. This occurs for triangles in regions IV or V. A rough upper bound in the number of steps is given by the sum 2 +

$\lceil 5.7\log(\frac{\pi}{6\sigma})\rceil + \lceil(\gamma-\pi/2)/\sigma\rceil$ This number is asymptotically linear in σ^{-1} because $\pi/3 \leq \gamma < \pi$.

2. For a triangle ABC above arc C_3, the number $N(ABC)$ of non-similar triangles is $1+N(ADC)$ (the 1 corresponds to $\triangle DBC$). The statement follows from Lemma 2, items 1, 2 and 3.

3. The complex case is region IV. (The analysis for region V is similar.) Here $N(ABC) = N(ADC) + N(CDE)$. First let us prove that $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$. If C is to the left of MO, then σ_{ADC} is the angle $\angle ADC$ and by Lemma 1 we are done. Next consider the geometric place of the set of points C such that $\beta_{ABC} = \frac{3}{2}\sigma_{ABC}$. This is a line L passing through D with negative slope. If C lies to the right of L, then $\triangle ADC$ will be in region IV to the left of MO and we are in the previous case in one step. If C lies in between L and MO, then $\sigma_{ADC} = \angle CAD = \beta_{ABC} \geq \frac{3}{2}\sigma_{ABC}$ by definition.

Now, using the fact that both triangles ADC and CDE have γ diminished by σ, the fact already proven that $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$, and observing that $\sigma_{DBC} \geq \sigma_{ABC}$, we have the following recurrence equation for the number $N(\gamma,\sigma)$ of non-similar triangles generated:

$$N(\gamma,\sigma) = N(\gamma-\sigma, \frac{3}{2}\sigma) + N(\gamma-\sigma,\sigma),$$

and Lemma 2.4 gives a bound to the number of necessary steps to take. It is not difficult to see that this recurrence essentially reduces to one of the type $f(n) = f(n/2) + f(n-1)$. This recurrence has no polynomial solution, and $\mathcal{O}(n^{\log n})$ is an upper bound, from where we get the bound $\mathcal{O}((\sigma^{-1})^{log(\sigma^{-1})}$.

It it interesting to note that not only the number of non-similar triangles generated by the bisection method is finite, but a stronger result can be proved:

Proposition 1. *The bisection method generates a finite number of different directions in the plane. Moreover, in the worst case this number is $\mathcal{O}(\sigma^{\sigma})$.*

Proof. Using Theorem 1, it is enough to show that in each step only finitely many new directions are added, and similar triangles generated use already generated directions. But we already know these facts from the analysis of the regions: at each step only one new direction is added except in regions IV and V where the number of directions is (possibly) doubled. Hence, a gross upper bound for the worst case is given by $\mathcal{O}(\sigma^{\sigma})$.

5 Classical results revisited

Using only elementary geometric methods it is possible to re-prove classical results about the smallest angle and parallel iterative bisection in the bisection method.

Theorem 2. *1. The bisection method gives $\mu_{ABC} \geq \frac{1}{2}\sigma_{ABC}$, where μ_{ABC} is the smallest angle in the mesh obtained by iteratively bisecting triangle ABC. For triangles below arc C_2 it holds that $\mu_{ABC} = \sigma_{ABC}$.*

2. *For each triangle, no more than 5 bisections (2 steps) are necessary in order to diminish the longest side (called diameter) by one half. For simultaneous parallel bisections of all triangles in the mesh, it holds $d_j \leq c2^{-j/2}d_0$ for a small constant c depending on the regions and d_j the diameter after j (parallel) bisections.*

Proof. 1. First, checking case by case it follows that for triangles in regions below C_2 always holds $\sigma_{ADC} > \sigma_{ABC}$ and $\sigma_{DBC} > \sigma_{ABC}$. Second, for triangles in region III, the new triangle ADC has $\sigma_{ADC} \geq \frac{3}{2}\sigma_{ABC}$ (because $\sigma_{ADC} = \angle ADC$ and Lemma 1), and clearly $\sigma_{DBC} > \sigma_{ABC}$. For triangles ABC in region II, observe that $\sigma_{ABC} \leq \pi/6$ and $\sigma_{ADC} = \angle ACD > \sigma_{ABC}$. Finally, once a triangle is in region I, we have Figure 5, being the worst case when $C = P$.

2. The first sentence is an easy observation, the worst case being triangles in region I.

As for the diameter bound, using formula the area of a triangle $A = \frac{1}{2}bh$ and the fact that the area decreases exactly by half after a bisection, one gets immediately $b_j = (\frac{h_0}{h_j})\frac{b_0}{2^j}$, where the sub-indexes indicate sides corresponding to a triangle in the j-th (parallel) bisection.

Now the key point is to observe that: (i) for triangles whose vertex C is below arcs C_2 or C_4 the diameter decreases by half after two parallel bisections, i.e. $d_2 \leq d_0/2$; and (ii) the fact we already know that, as bisection progresses, triangles go "up" the level of arcs C_4 and C_2. Hence, h_j can be bound (in terms of b_j) because from the fact mentioned above that we can deduce that σ_j is no smaller than say $\pi/7$. Similarly, h_0 has a fixed bound in terms of b_0 (the worst case being $\sqrt{3}b_0/2$). Using these formulas we get $b_j^2 \leq c^2 b_0^2 2^{-j}$, for some constant $c \leq \sqrt{3}$ (cf. also [1]). From here, taking square root we get the statement of the theorem.

6 Conclusion

We presented a taxonomy of triangles in the plane which captures the behavior of the bisection method. Besides allowing us to prove complexity results for the bisection method, this classification is useful to refine bounds for each class of triangles, and to determine more precisely lower bounds on the smallest angle μ_{ABC} in the mesh, as well as the number of non-similar triangles generated. The analysis could be further refined considering regions we did not separate, e.g. below arc C_2, above arc C_3 and to the left of MO in Figure 4. Further work includes use of this theoretical analysis to refine algorithms of bisection (4-edge partition, simple bisection, etc.) according to the type of triangle found in each iteration.

References

1. A. Adler, *On the Bisection Method for Triangles*, Mathematics of Computation, Vol. 40, Number 162, April 1983, pp. 571-574.

2. B. Kearfott, *A Proof of Convergence and an Error Bound for the Method of Bisection in R^n*, Mathematics of Computation, Vol. 32, Number 144, October 1978, pp. 1147-1153.
3. J. O'Rourke, *Computational Geometry Column 23*, International Journal of Computational Geometry & Applications, Vol. 4, No. 2 (1994), pp. 239-242.
4. M.C. Rivara, *Algorithms for refining triangular grids suitable for adaptive and multigrid techniques*, International journal for numerical methods in Engineering, vol.20, pp. 745-756, 1984.
5. M-C. Rivara, G. Irribarren, *The 4-Triangles Longest-side Partition of Triangles and Linear Refinement Algorithms*, Mathematics of Computation, Vol. 65, Number 216, October 1996, pp. 1485-1502.
6. M.C. Rivara, C. Levin, *A 3d Refinement Algorithm for adaptive and multigrid Techniques*, Communications in Applied Numerical Methods, vol. 8, pp. 281-290, 1992.
7. I.G. Rosenberg, F. Stenger, *A Lower Bound on the Angles of Triangles Constructed by Bisecting the Longest Side*, Mathematics of Computation, Vol. 29, Number 130, April 1975, pp. 390-395.
8. M. Stynes, *On Faster Convergence of the Bisection Method for certain Triangles*, Mathematics of Computation, Vol. 33, 1979, pp. 1195-1202.
9. M. Stynes, *On Faster Convergence of the Bisection Method for all Triangles*, Mathematics of Computation, Vol. 35, Number 152, October 1980, pp. 1195-1202.