Non-Interactive Zero-Knowledge Proofs: Shuffles

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UNIVERSIDAD DE CHILE

Joint work with Carla Ràfols and Alejandro Hevia

1 Introduction
- Mix-nets and Shuffles
- Non-Interactive Zero-Knowledge
- Bilinear Groups and Tools

2 State of the art
- Groth-Sahai Proofs
- NIZK for membership in Linear Subspaces of $\hat{\mathbb{G}}^n$

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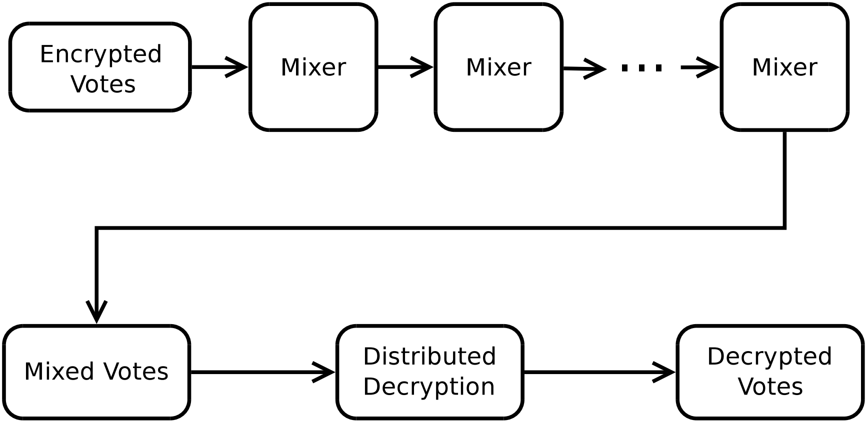
- NIZK for membership in Linear Subspaces of $\hat{\mathbb{G}}^m \times \check{\mathbb{H}}^n$
- Aggregation of quadratic equations

4 Efficient NIZK Shuffle Arguments

Mix-net based voting scheme



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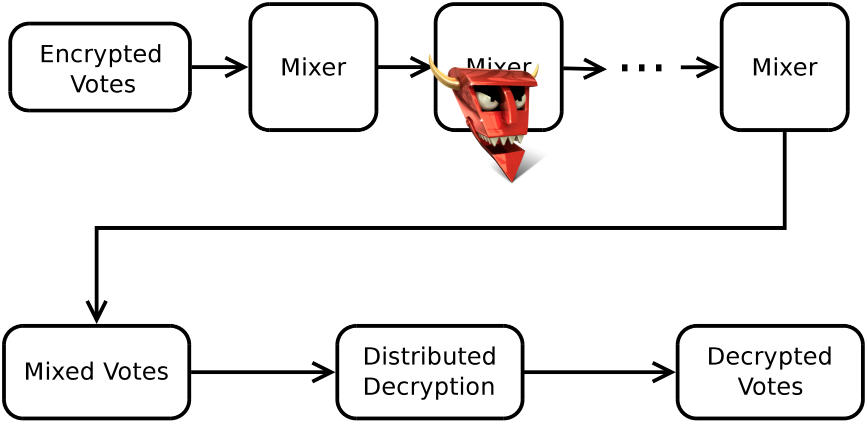

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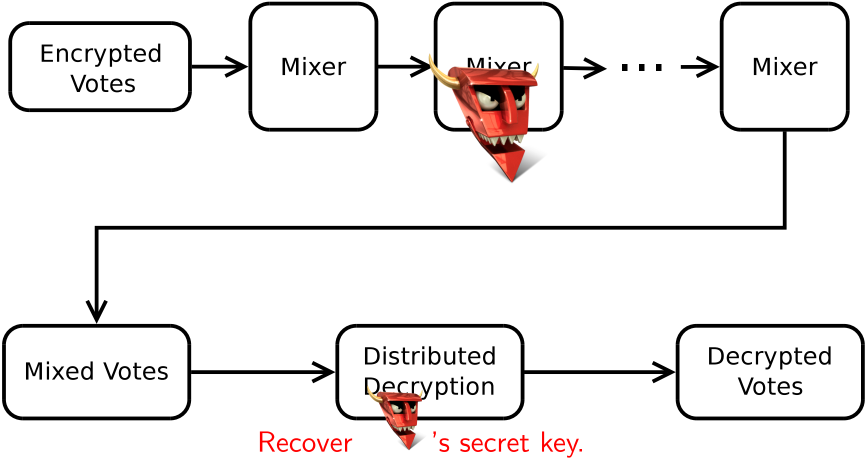

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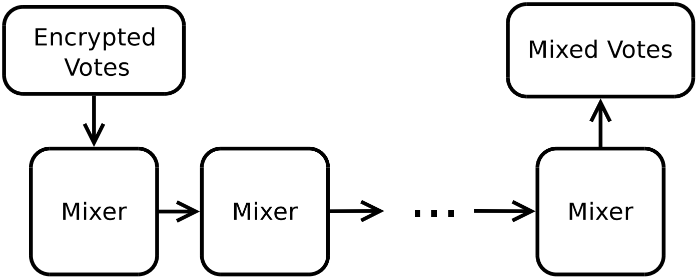

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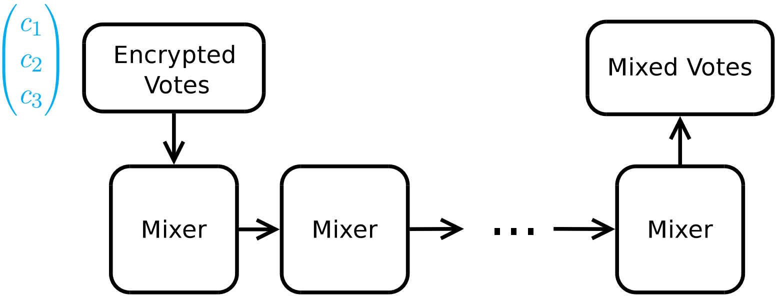
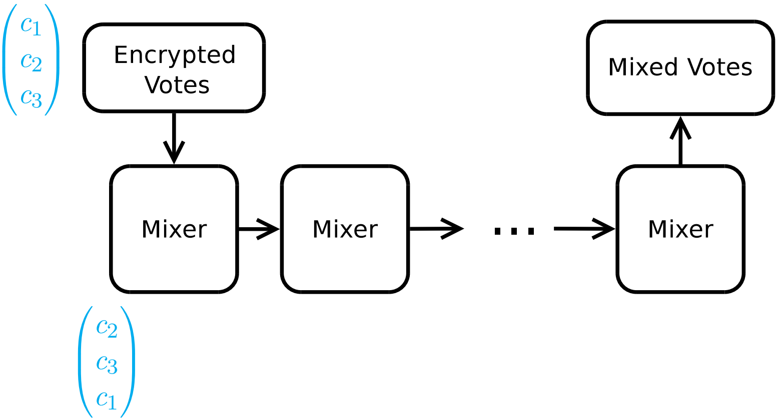


$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$
Encrypted Votes
Mixer
Mixer
...
Mixer
Mixed Votes

Mix-nets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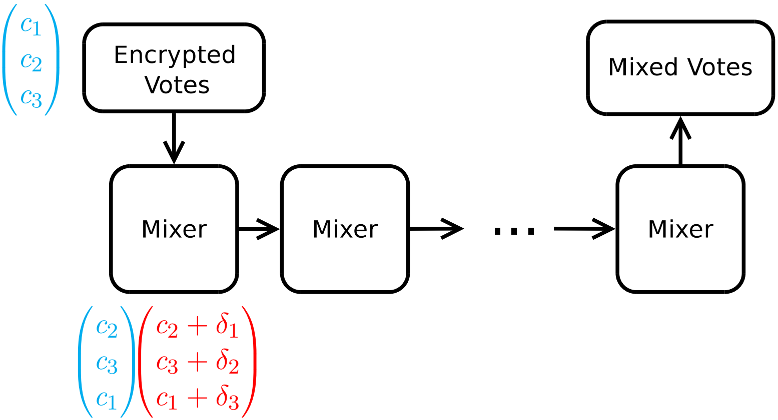

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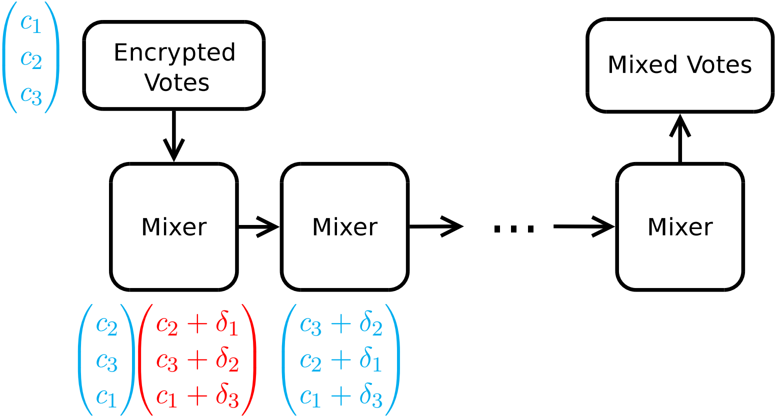
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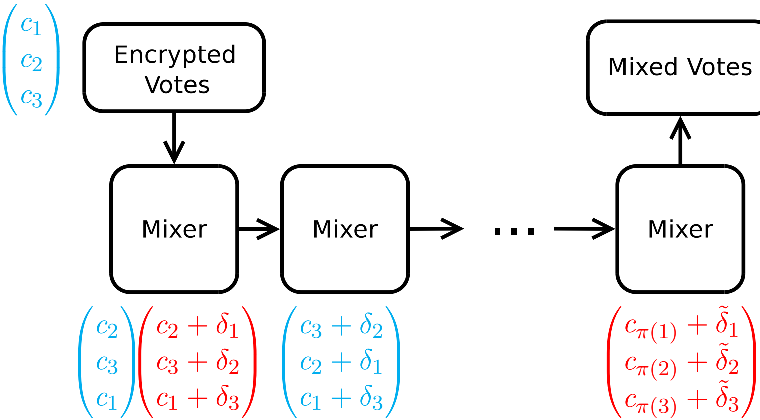
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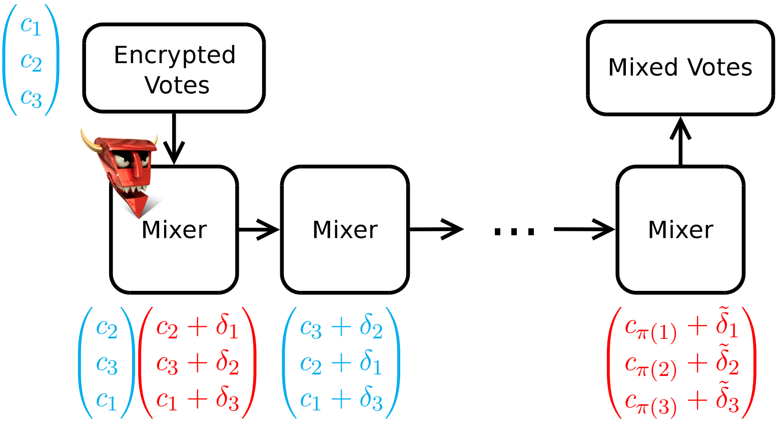
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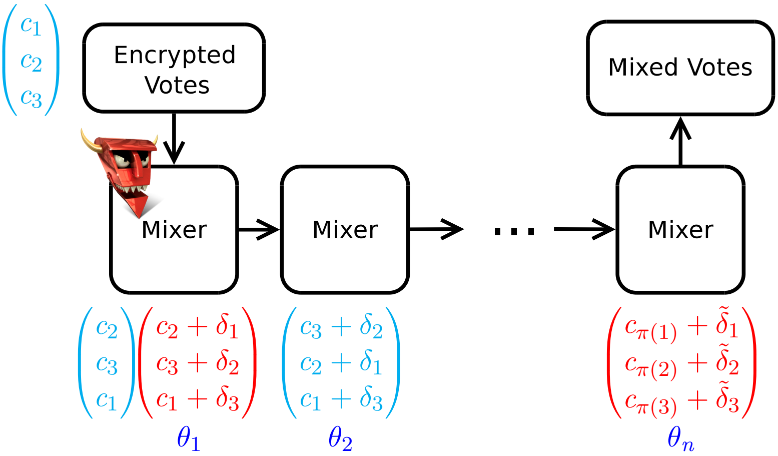
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Zero-Knowledge Shuffle Argument

We want to prove that each $\mathbf{c}, \mathbf{d}$ belongs to

$$L_{\textsf{shuffle}} := \{(\mathbf{c}, \mathbf{d}) : \exists \pi \in S_n \text{ s.t. } \forall i \in [n]\ c_i - d_{\pi(i)} \text{ is an encryption of } 0\}.$$

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Non-Interactive Zero-Knowledge Proofs for $L \in \text{NP}$

(x, w)
$\downarrow$

Peggy

x
$\downarrow$

Victor

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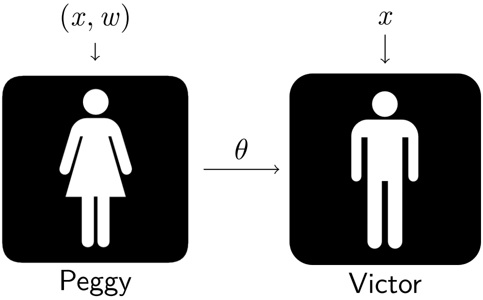

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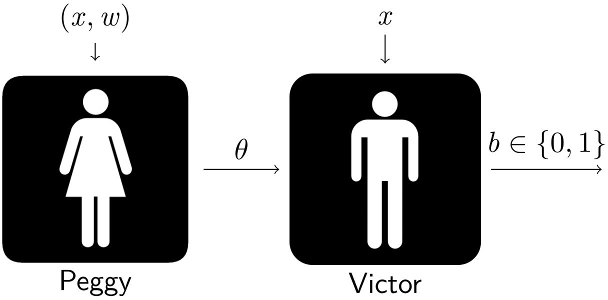

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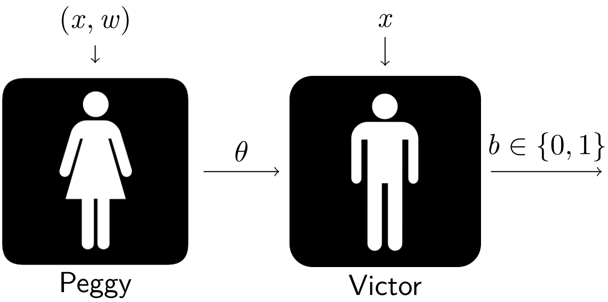

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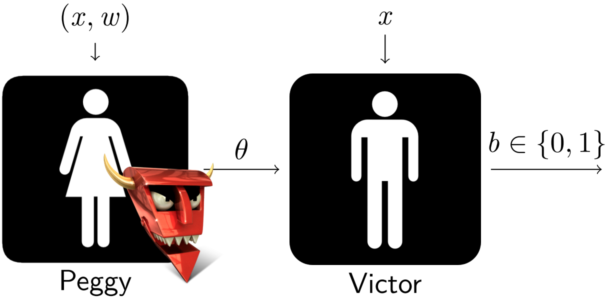

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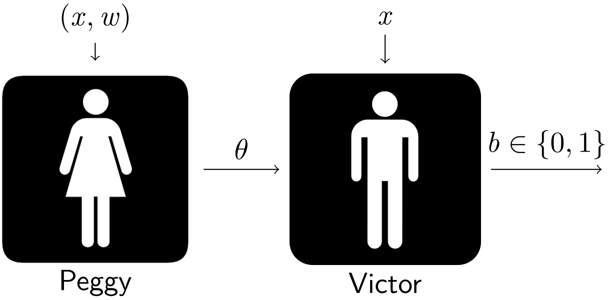

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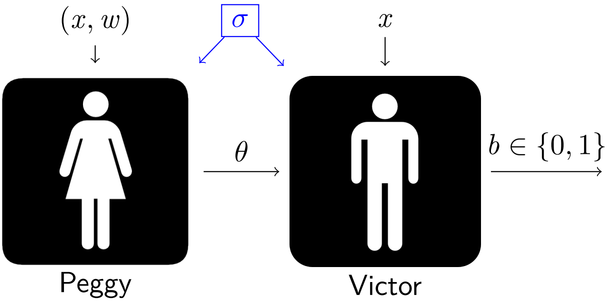

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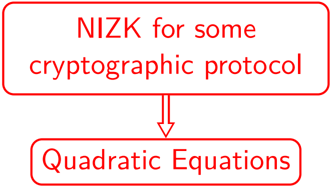

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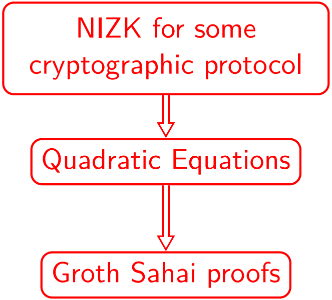

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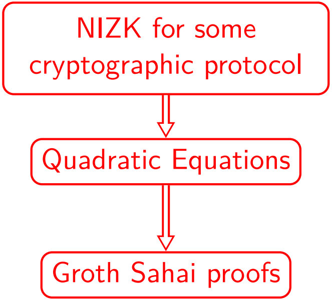

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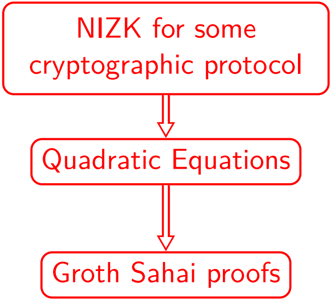

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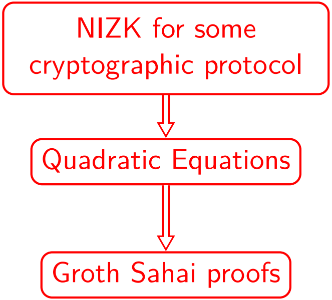

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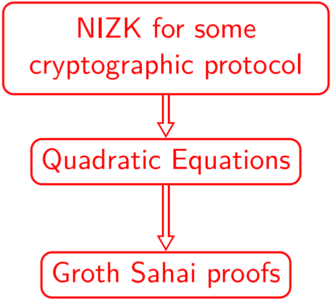

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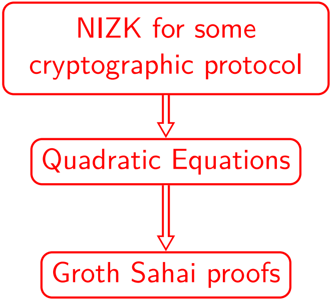

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- Previous work: Optimize GS proofs to $O(m)$ for other linear equations and quadratic equations.

Bilinear Groups

3 additive cyclic groups $\hat{\mathbb{G}}, \check{\mathbb{H}}$ and $\mathbb{T}$ with a bilinear map or pairing

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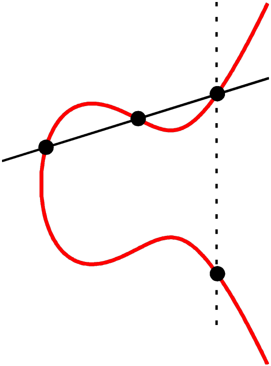

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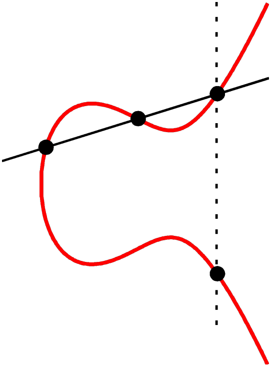

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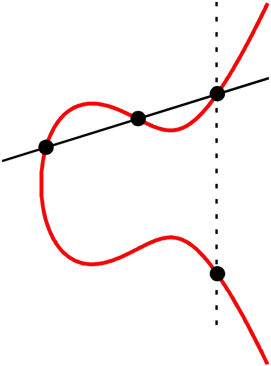

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- $\mathcal{A}$ wins iff $b' = b$.

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Decryption: The row vector $(1, -u_{2,1}) \in \mathbb{Z}_q^{1\times 2}$ allows to recover $\hat{m}$.

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1 Introduction
- Mix-nets and Shuffles
- Non-Interactive Zero-Knowledge
- Bilinear Groups and Tools

2 State of the art
- Groth-Sahai Proofs
- NIZK for membership in Linear Subspaces of $\hat{\mathbb{G}}^n$

3 Prior Work
- NIZK for membership in Linear Subspaces of $\hat{\mathbb{G}}^m \times \check{\mathbb{H}}^n$
- Aggregation of quadratic equations

4 Efficient NIZK Shuffle Arguments

Groth-Sahai Proofs [GS08]

Groth-Sahai (GS) Proofs are NIZK proofs for the satisfiability of equations of the form

$$\sum_{j\in[m_y]} \hat{\alpha}_j \check{y}_j + \sum_{i\in[m_x]} \hat{x}_i, \check{\beta}_i + \sum_{i\in[m_x]} \sum_{j\in[m_y]} \gamma_{i,j} \hat{x}_i \check{y}_j = t, \tag{PPE}$$

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While the CRS size is $|ck| = O(1)$.

Proofs of Membership in Linear Sub-spaces of $\hat{\mathbb{G}}^n$

Observation 1

Ciphertexts $\hat{\mathbf{c}} := \hat{w}_1\mathbf{e}_1 + r_1\hat{\mathbf{u}}$ and $\hat{\mathbf{d}} := \hat{w}_2\mathbf{e}_1 + r_2\hat{\mathbf{u}}$ open to the same value iff there exists some $r \in \mathbb{Z}_q$ s.t. $\hat{\mathbf{c}} - \hat{\mathbf{d}} = r\hat{\mathbf{u}}$.

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Kernel Assumptions

The security of the constructions for Linear Subspaces can be based on the next assumption.

Definition (Simultaneos Pairing Assumption)

Any adversary $\mathcal{A}$ has at most negligible probability of winning in the next game:

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- How to prove that commitment $\check{\mathbf{f}}_1, \ldots, \check{\mathbf{f}}_n$ to $\check{\mathbf{b}}_\Delta$ was correctly computed?

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- Groth and Lu's construction is the most efficient construction under mild assumptions with $O(n)$ communication.

Proving that a commitment opens to a permutation

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- Note that $\sum_i \check{\mathbf{f}}_i = \sum_i \ell_i \check{\mathbf{a}}_{\Delta,i}$ for $\ell_i \in \mathbb{Z}_n$.
- If there is some $\ell_i \neq 1$ then $(\ell_1 - 1, \ldots, \ell_n - 1) \in \mathbf{Ker}(\check{\mathbf{a}}_\Delta^\top)$.

GS aggregation and QA-NIZK in asymmetric groups

We construct constant-size QA-NIZK proofs of membership in the language

$$L_{\hat{\mathbf{M}},\check{\mathbf{N}}} = \left\{ (\hat{\mathbf{x}}, \check{\mathbf{y}}) \in (\hat{\mathbb{G}}^m \times \check{\mathbb{H}}^n) : \exists \mathbf{w} \in \mathbb{Z}_q^t \text{ s.t. } \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} \mathbf{w} \right\}.$$

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- Similar techniques allows to aggregate the proof of n two-sided linear equations into only two GS proofs.

Aggregation of Quadratic equations over $\mathbb{Z}_q$

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$$L_{\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2,\mathsf{bits}} = \{\hat{\mathbf{c}} \in \hat{\mathbb{G}}^n : \exists \mathbf{b} \in \{0,1\}^n, \mathbf{w} \in \mathbb{Z}_q^m \text{ s.t } \hat{\mathbf{c}} = \hat{\mathbf{U}}_1\mathbf{b} + \hat{\mathbf{U}}_2\mathbf{w}\}.$$

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No such construction was known even in Symmetric Groups!

Higher degree equations

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- Proof for l equations can be aggregated into a single $O(n)$ proof.

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- We reviewed NIZK proofs of membership in linear subspaces.
- We reviewed aggregation of quadratic equations.
- We showed how to construct efficient NIZK Shuffle Arguments under mild assumptions.

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- Non-Interactive Zero-Knowledge
- Bilinear Groups and Tools

2 State of the art
- Groth-Sahai Proofs
- NIZK for membership in Linear Subspaces of $\hat{\mathbb{G}}^n$

3 Prior Work
- NIZK for membership in Linear Subspaces of $\hat{\mathbb{G}}^m \times \check{\mathbb{H}}^n$
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4 Efficient NIZK Shuffle Arguments

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